Local Surface Theory and Gauss Curvature

Let $S : U \to \mathbb{R}^3$ be a "surface patch", that is, with $S(u,v) = (x(u,v), y(u,v), z(u,v))$ and

$S_u = \left(\frac{\partial x(u,v)}{\partial u}, \frac{\partial y(u,v)}{\partial u}, \frac{\partial z(u,v)}{\partial u}\right)$ and

$S_v = \left(\frac{\partial x(u,v)}{\partial v}, \frac{\partial y(u,v)}{\partial v}, \frac{\partial z(u,v)}{\partial v}\right)$, we require that

$S_u \times S_v \neq \vec{0}$. Define the unit normal by

$\vec{N} = (S_u \times S_v)/\|S_u \times S_v\|$. $N = N(u,v)$ depends on u and v.

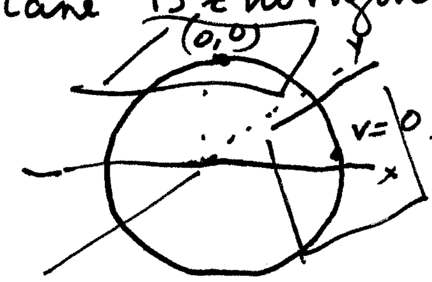
Definition: The plane generated by S_u and S_v is called the <u>tangent</u> <u>plane</u> of S (at a given (u,v) value)

Example: If $S(u,v) = (u, v, \sqrt{1-u^2-v^2})$ then the tangent plane is the plane generated by $(1, 0, -u/\sqrt{\ \ })$ and $(0, 1, -v/\sqrt{\ \ })$. The unit normal is

$(1, 0, -u/\sqrt{\ \ }) \times (0, 1, -v/\sqrt{\ \ })/\| \quad \| =$

$\left(\frac{u}{\sqrt{\ \ }}, \frac{v}{\sqrt{\ \ }}, 1\right)/\| \quad \| = (u, v, \sqrt{1-u^2-v^2}) = \vec{N}(u,v)$

(Check: This is unit length & $\perp$ to S_u and S_v). This coincides with geometric intuition: the position vector is normal to the sphere around the origin.) Note that as $u^2+v^2 \to 1^-$ the normal approaches lying in the (x,y) plane so the tangent plane becomes nearly vertical, i.e., approaches containing $(0,0,1)$. When $(u,v) = (0,0)$, the normal is $(0,0,1)$: the tangent plane is horizontal.



The patch here is defined (& nonsingular) only when $u^2+v^2 < 1$, i.e. $U = \{(u,v) \in \mathbb{R}^2 : u^2+v^2 < 1\}$

We are interested in how $N(u,v)$ varies. This interest arises from noting that change in N corresponds intuitively to S being curved. The extreme case of this intuition is made formal by this result:

Theorem: If $U \subset \mathbb{R}^2$ is connected and $S: U \to \mathbb{R}^3$ is a patch (with $S_u \times S_v \neq \vec{0}$), and if $N(u,v)$ is constant, then $S(U)$ lies in a plane in $\mathbb{R}^3$.

Proof: Suppose $N(u,v) = \underset{\text{constant}}{N_0}$ for all u,v in U.

Then $\frac{\partial}{\partial u}\langle S, N_0\rangle = \langle S_u, N_0\rangle$ since $\frac{\partial}{\partial u}N_0 = 0$.

$= 0$ since S_u is $\perp$ to $N(u,v) = N_0$

Also, by the same reasoning $\frac{\partial}{\partial v}\langle S, N_0\rangle = 0$. Hence $\langle S(u,v), N_0\rangle$ is a constant c so $S(u,v)$ lies in the plane $\{\vec{v} : \langle \vec{v}, N_0\rangle = c\}$. $\square$

In this argument, we used the Leibnitz Rule for differentiation of inner products $\langle , \rangle$ that we used earlier in curve theory.

Now $\langle N, N\rangle \equiv 1$ so $\langle \frac{\partial N}{\partial u}, N\rangle = 0$ and $\langle \frac{\partial N}{\partial v}, N\rangle = 0$. Thus the derivative $\frac{\partial N}{\partial u}$ and $\frac{\partial N}{\partial v}$ lie in the tangent plane: they are $\perp$ to N!

Moreover, $\langle \frac{\partial N}{\partial u}, S_u\rangle = -\langle S_{uu}, N\rangle$

and $\langle \frac{\partial N}{\partial u}, S_v\rangle = -\langle S_{vu}, N\rangle$ ← these two are equal

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and $\langle \frac{\partial N}{\partial v}, S_u\rangle = -\langle S_{uv}, N\rangle$

and $\langle \frac{\partial N}{\partial v}, S_v\rangle = -\langle S_{vv}, N\rangle$

These formulas come from differentiating
$\langle S_u, N\rangle = 0$ and $\langle S_v, N\rangle = 0$!

Now S_{uu}, for example, can have both "tangent and normal parts", that is it can, when expressed as $S_{uu} = \alpha N + (\text{vector in tangent plane})$, it can have both parts nonzero.

Notation:

$$L_{11} = -\langle S_{uu}, N\rangle = +\langle S_u, \tfrac{\partial N}{\partial u}\rangle$$
$$L_{12} = -\langle S_{uv}, N\rangle = \langle S_u, \tfrac{\partial N}{\partial v}\rangle = \langle S_v, \tfrac{\partial N}{\partial u}\rangle$$
$$L_{22} = -\langle S_{vv}, N\rangle = \langle S_v, \tfrac{\partial N}{\partial v}\rangle .$$

So the tangent part of $S_{uu} = S_{uu} - \langle S_{uu}, N\rangle N$ (since N is a unit vector!) and this $= S_{uu} + L_{11}N$.
Similarly tang part$(S_{uv}) = S_{uv} + L_{12}N$ and tang part$(S_{vv}) = S_{vv} + L_{22}N$. We think of the L's as a matrix $\begin{pmatrix} L_{11} & L_{12} \\ L_{12} & L_{22}\end{pmatrix}$, which is symmetric ("$L_{21} = L_{12}$" in an obvious notation expressing $\langle S_{uv}, N\rangle = \langle S_{vu}, N\rangle$).

For temporary but useful notation, write $T(\ ) =$ tangent part of whatever is in the ( ) parentheses.
Similarly, we write $N(\ )$ for the normal part.

We also introduce the (permanent) notations
$E = \langle S_u, S_u\rangle$, $F = \langle S_u, S_v\rangle$ and $G = \langle S_v, S_v\rangle$

The quantities (functions of u & v) E, F, G are of interest because they specify the "geometry of S": in particular, if $\gamma(t) = (u(t), v(t))$ is a curve in U, then the arclength of the curve $S(\gamma(t))$ in $\mathbb{R}^3$ is given by $\int \sqrt{E\left(\frac{du(t)}{dt}\right)^2 + 2F\left(\frac{du(t)}{dt}\frac{dv(t)}{dt}\right) + G\left(\frac{dv(t)}{dt}\right)^2}$ since the tangent vector of $S(\gamma(t))$ is (by the Chain Rule) $\vec{S}_u \frac{du(t)}{dt} + \vec{S}_v \frac{dv(t)}{dt}$ (or, putting the numbers in front of the vectors to be more conventional, $\frac{du(t)}{dt}\vec{S}_u + \frac{dv(t)}{dt}\vec{S}_v$).

Example: Consider the patch $S(r,\theta)$, $0 < r < \pi$, $-\pi < \theta < \pi$, $S(r,\theta) = (\sin r \cos\theta, \sin r \sin\theta, \cos r)$. Note that this is (part of a) sphere, namely, the sphere of radius 1 around $\vec{0}$ because

$$(\sin r \cos\theta)^2 + (\sin r \sin\theta)^2 + \cos^2 r = \sin^2 r(\cos^2\theta + \sin^2\theta) + \cos^2 r = 1.$$

We compute E, F, and G for this patch.

$$S_r = (\cos r \cos\theta, \cos r \sin\theta, -\sin r)$$
$$S_\theta = (-\sin r \sin\theta, \sin r \cos\theta, 0)$$

So
$$E = \langle S_r, S_r\rangle = \cos^2 r \cos^2\theta + \cos^2 r \sin^2\theta + (\sin r)^2 = 1$$
$$F = \langle S_r, S_\theta\rangle = -\cos r \sin r \cos\theta \sin\theta + \cos r \sin r \sin\theta \cos\theta = 0$$
$$G = \langle S_\theta, S_\theta\rangle = (-\sin r \sin\theta)^2 + (\sin r \cos\theta)^2 = (\sin^2 r)(\sin^2\theta + \cos^2\theta) = \sin^2 r.$$

Let's compute L_{11}, L_{12}, and L_{22} also:

Since $N = (\sin r\cos\theta, \sin r\sin\theta, \cos r)$ (Normal = position vector for sphere, or compute $S_r \times S_\theta / \| \ \|$)

$L_{11} = -\langle S_{rr}, N\rangle$

$= -\langle \frac{\partial}{\partial r}(\cos r\cos\theta, \cos r\sin\theta, -\sin r), N\rangle$

$= -\langle(-\sin r\cos\theta, -\sin r\sin\theta, -\cos r), (\sin r\cos\theta, \sin r\sin\theta, \cos r)\rangle$

$= +1.$

Similarly, $L_{12} = -\langle S_{r\theta}, N\rangle = \langle(-\cos r\sin\theta, \cos r\cos\theta, 0), N\rangle$

$= 0$ and $L_{22} = -\langle S_{\theta\theta}, N\rangle = -\langle(-\sin r\cos\theta, -\sin r\sin\theta, 0), N\rangle$

$= +\sin^2 r.$

We shall see later that there is some serious geometric meaning to the fact that $(L_{11}L_{22} - L_{12}^2)/(EG - F^2) = 1$!

Returning now to general considerations and recalling our "tangent part" notation $T(\ )$, we want to establish a remarkable fact first observed by Gauss: the "tangent parts" $T(S_{uu})$, $T(S_{uv})$ and $T(S_{vv})$ can be computed from E, F, and G (and their derivatives). Let us try $T(S_{uu})$ first.

Now $\langle T(S_{uu}), S_u\rangle = \langle S_{uu}, u\rangle$ since $\langle N(S_{uu}), S_u\rangle = 0$ and of course $S_{uu} = T(S_{uu}) + N(S_{uu})$ where $N =$ the normal part as before. But $\langle S_{uu}, S_u\rangle$

$= \frac{1}{2}\frac{\partial}{\partial u}\langle S_u, S_u\rangle = \frac{1}{2}\frac{\partial E}{\partial u} = \frac{1}{2}E_u$, the latter equality being just a convenient notation now.

What about $\langle S_{uu}, S_v\rangle = \langle T(S_{uu}), S_v\rangle$?

$\langle S_{uu}, S_v\rangle = \frac{1}{2}\left(2\frac{\partial}{\partial u}\langle S_u, S_v\rangle - \frac{\partial}{\partial v}\langle S_u, S_u\rangle\right).$

<u>To Check:</u> $2\frac{\partial}{\partial u}\langle S_u, S_v\rangle = 2\langle S_{uu}, S_v\rangle + 2\langle S_u, S_{uv}\rangle$

$\frac{\partial}{\partial v}\langle S_u, S_u\rangle = 2\langle S_u, S_{uv}\rangle$. So works.

So $\langle S_{uu}, S_u \rangle = \frac{1}{2} E_u$
$\langle S_{uu}, S_v \rangle = F_u - \frac{1}{2} E_v$. $\Bigg\}(*)$

From this, we can compute $T(S_{uu})$ by linear algebra (of a simple kind: two equations in two unknowns!) Namely, suppose, as must be true, that
$T(S_{uu}) = a S_u + b S_v$ for some a, b $\mathbb{R}$-valued.

Then $\langle S_{uu}, S_u \rangle = a \langle S_u, S_u \rangle + b \langle S_u, S_v \rangle$
$= aE + bF$

and $\langle S_{uu}, S_v \rangle = aF + bG$.

Then $aE + bF = \frac{1}{2} E_u$ and $aF + bG = F_u - \frac{1}{2} E_v$.

We can solve these for a and b (in terms of E, F, G and their derivatives) because $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ is an invertible matrix. And why is $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ invertible? Because $EG - F^2 > 0$ by the Cauchy Schwarz Inequality! (since S_u and S_v are linearly independent).

The corresponding to $(*)$ items for S_{uv} and S_{vv} are

$\langle S_{uv}, S_u \rangle = \frac{1}{2}\left(\frac{\partial}{\partial v} \langle S_u, S_u \rangle \right) = \frac{1}{2} E_v$
$\langle S_{uv}, S_v \rangle = \frac{1}{2} \frac{\partial}{\partial u} \langle S_v, S_v \rangle = \frac{1}{2} G_u$
$\langle S_{vv}, S_u \rangle = F_v - \frac{1}{2} G_u$
$\langle S_{vv}, S_v \rangle = \frac{1}{2} G_v$

So, if, following Gauss, we call items "intrinsic" if they can be computed from E, F, G and their derivatives (of any orders), then we have shown that $T(S_{uu})$, $T(S_{uv})$, and $T(S_{vv})$ are intrinsic

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Continuing our spherical patch $S(r,\theta) = (\sin r\cos\theta, \sin r\sin\theta, \cos r)$ example, let us compute $T(S_{\theta\theta})$ "intrinsically":

$$\langle T(S_{\theta\theta}), S_\theta\rangle = \frac{1}{2}\frac{\partial}{\partial\theta}\langle S_\theta, S_\theta\rangle = \frac{1}{2}\frac{\partial}{\partial\theta}(\sin^2 r) = 0$$

$$\langle T(S_{\theta\theta}), S_r\rangle = \frac{1}{2}\left(2\frac{\partial}{\partial\theta}\langle S_\theta, S_r\rangle - \frac{\partial}{\partial r}\langle S_\theta, S_\theta\rangle\right)$$

$$= \frac{1}{2}\left(0 - \frac{\partial}{\partial r}\sin^2 r\right) = -\sin r\cos r$$

Since $T(S_{\theta\theta}) = a\,S_r + b\,S_\theta$ necessarily for some a, b,

$\langle T(S_{\theta\theta}), S_r\rangle = a = -\sin r\cos r$, $\langle T(S_{\theta\theta}), S_\theta\rangle = 0 = b\sin^2 r$

So $T(S_{\theta\theta}) = -\sin r\cos r\, S_r$.

You can check this directly: $S_{\theta\theta} = (-\sin r\cos\theta, -\sin r\sin\theta, 0)$

so $\langle S_{\theta\theta}, N\rangle = -\sin^2 r$. So normal part of $S_{\theta\theta} = (-\sin^2 r)N$

$$T(S_{\theta\theta}) = S_{\theta\theta} - \text{normal part}$$

$$= (-\sin r\cos\theta, -\sin r\sin\theta, 0) + \sin^2 r(\sin r\cos\theta, \sin r\sin\theta, \cos r)$$

$$= \left(\sin r\overbrace{(-1+\sin^2 r)}^{\cos^2}\cos\theta,\ \sin\theta(\sin r)\overbrace{(-1+\sin^2 r)}^{\cos^2},\ \cos r\sin^2 r\right)$$

$$= (-\sin r\cos r)(\cos r\cos\theta, \cos r\sin\theta, -\sin r)$$

$$= (-\sin r\cos r)S_r \text{ as we found intrinsically!}$$

Returning to generalities:

We all know examples of surfaces that are not pieces of a plane and so are curved in that sense, but that are not curved intrinsically: just think of a piece of paper made by bending into a half-cylinder. Intrinsically, it is still a piece of $\mathbb{R}^2$: flat, uncurved — the making it cylindrical did not change the length of curves on it. We can make this explicit: set $S(u,v) = (u, \cos v, \sin v)$ $-\infty < u < +\infty$, $-\frac{\pi}{2} < v < \frac{\pi}{2}$, for example. Then $S_u = (1,0,0)$, $S_v = (0, -\sin v, \cos v)$. So $E = 1$, $F = 0$, $G = 1$, the same E, F, and G as for the "planar embedding" $(u,v) \to (u,v,0)$.

On the other hand, intuitively it seems that the sphere is curved intrinsically: we ought to be able to compute just in terms of E, F, and G that it is not like a plane. (Otherwise, we could make distance-perfect maps of at least pieces of the earth's surface, and we all have the feeling from experience that this doesn't work: we cannot flatten a piece of paper on a basketball without crumpling it some).

To see how to try to identify "intrinsic" curvature, in this intuitive sense (we'll make this precise soon), let us look first at another example:

Let $S(u,v) = (u, v, f(u,v))$ where $f(0,0)=0$ and $f_u(0,0) = 0 = f_v(0,0)$. Actually, any surface can be written like this if you translate it and rotate it (Exercise for you). Then $S_u = (1,0,f_u)$, $S_v = (0,1,f_v)$, $N(0,0) = (0,0,1)$, $L_{11} = -\langle S_{uu}, N\rangle$ so

$L_{11}(0,0) = -\langle (0,0,f_{uu}), (0,0,1)\rangle = -f_{uu}$

$L_{12}(0,0) = -\langle (0,0,f_{uv}), (0,0,1)\rangle = -f_{uv}$

$L_{22} = -\langle (0,0,f_{vv}), (0,0,1)\rangle = -f_{vv}$,

$E = (1+f_u^2)$, $F = f_u f_v$, $G = (1+f_v^2)$. So at $(0,0)$, $E=1$, $F=0$, $G=1$ and <u>all</u> <u>first</u> <u>derivatives</u> of E, F, G vanish (at $(0,0)$). So we can see that S is not part of a plane at $(0,0)$ if some second derivative of f at $(0,0)$ is nonzero by looking at the L's (normal parts of $S_{\cdot\cdot}$). But we cannot see that S is curved from looking at the tangent parts of $S_{\cdot\cdot}$ items — two derivatives — $T(S_{uu})$ etc. — at $(0,0)$, even if f does have some nonzero second derivatives, Look at formulas <u>for</u> $S_{\cdot\cdot}$ to see why! Thinking about this for a while suggests that to see "intrinsic curvature" — assuming it exists(!) — we ought to look at third order S derivatives: things like S_{uvu} and so on. We need this motivating example because things are about to get a little messy computationally. But we have nowhere else to go, as the example shows.

Gauss had a lot of determination, and eventually he figured out that he could get somewhere with this third derivative business. Here is one way it can be made to work: Let's start off by noticing that $(S_{uv})_u = (S_{uu})_v$. This is just an aspect of equality of mixed partials. So (this is just what works!)

$< (S_{uv})_u, S_v > = < (S_{uu})_v, S_v >$. Now

$$< (S_{uv})_u, S_v > = \frac{\partial}{\partial u} < S_{uv}, S_v > - < S_{uv}, S_{uv} >$$

$$= \frac{\partial}{\partial u} < S_{uv}, S_v > - < T(S_{uv}), T(S_{uv}) > - < N(S_{uv}), N(S_{uv}) >$$

since $< S_{uv}, S_{uv} > = < T(S_{uv}) + N(S_{uv}), T(S_{uv}) + N(S_{uv}) >$ and the mixed N & T terms give 0.

Similarly

$$< (S_{uu})_v, S_v > = \frac{\partial}{\partial v} < S_{uu}, S_v > - < S_{uu}, S_{vv} >$$

$$= \frac{\partial}{\partial v} < S_{uu}, S_v > - < T(S_{uu}), T(S_{vv}) > - < N(S_{uu}), N(S_{vv}) >$$

Now $< N(S_{uv}), N(S_{uv}) > = L_{12}^2$ while $< N(S_{uu}), N(S_{vv}) > = L_{11}L_{22}$

Since $< (S_{uv})_u, S_v > - < (S_{uu})_v, S_v > = 0$, combining all this we get

$$L_{11}L_{22} - L_{12}^2 = -\frac{\partial}{\partial u} < S_{uv}, S_v > + < T(S_{uv}), T(S_{uv}) >$$
$$+ \frac{\partial}{\partial v} < S_{uu}, S_v > - < T(S_{uu}), T(S_{vv}) >$$

Why is this interesting? Because the right hand side is intrinsic(!) since $< S_{uu}, S_v > = < T(S_{uu}), S_v >$ and $< S_{uv}, S_v > = < T(S_{uv}), S_u >$ are, and so are the other two terms

This is already impressive. But, while the right hand side is intrinsic in the sense of being computable in terms of E, F and G and their derivatives with respect to u and v, the right-hand side (and hence the lefthand side) does depend on the choice of "coordinates u and v. For example, if we doubled u and v, then the right-hand side would be multiplied by 16.

What we would like most of all is something that is not only ~~[illegible]~~ intrinsic but also "invariant" – independent of choice of u and v, too, unchanging under reparameterization. Such an item lies close at hand:

Definition/Lemma: $(L_{11}L_{22}-L_{12}^2)/(EG-F^2)$ a number! is called the Gauss curvature: it is intrinsic and coordinate-choice independent, depending only on the chosen point of the surface.

Proof the Lemma part: We have disposed of the "intrinsic" part. To get the "invariant" part, suppose $\hat{u}, \hat{v}$ are another coordinate system at the same point. Then N and $\hat{N}$ are the same (up to $\pm$: by interchanging $\hat{u}$ and $\hat{v}$, which leaves the $(LL-L^2)/(EG-F^2)$ item unchanged, we can assume $N = +\hat{N}$). Then since $\hat{L}_{11} = -\langle S_{\hat{u}\hat{u}}, \hat{N}\rangle = \langle S_{\hat{u}}, N_{\hat{u}}\rangle$

and similarly for $\hat{L}_{12}$, $\hat{L}_{22}$ we get from the Chain Rule:

$$\hat{L}_{11} = \langle S_u \frac{\partial u}{\partial \hat{u}} + S_v \frac{\partial v}{\partial \hat{u}}, \; N_u \frac{\partial u}{\partial \hat{u}} + N_v \frac{\partial v}{\partial \hat{u}} \rangle$$

$$= \left(\frac{\partial u}{\partial \hat{u}}\right)^2 L_{11} + 2 \frac{\partial v}{\partial \hat{u}} \frac{\partial u}{\partial \hat{u}} L_{12} + \left(\frac{\partial v}{\partial \hat{u}}\right)^2 L_{22}$$

and similar formulas for $\hat{L}_{12}$ and $\hat{L}_{22}$. Tedious but routine calculation gives

$$\hat{L}_{11}\hat{L}_{22} - \hat{L}_{12}^2 = (L_{11}L_{22} - L_{12}^2)\left(\left(\frac{\partial u}{\partial \hat{u}}\right)\left(\frac{\partial v}{\partial \hat{v}}\right) - \left(\frac{\partial u}{\partial \hat{v}}\right)\left(\frac{\partial v}{\partial \hat{u}}\right)\right)^2$$

A similar calculation gives

$$\hat{E}\hat{G} - \hat{F}^2 = (EG - F^2)\left(\left(\frac{\partial u}{\partial \hat{u}}\right)\left(\frac{\partial v}{\partial \hat{v}}\right) - \left(\frac{\partial u}{\partial \hat{v}}\right)\left(\frac{\partial v}{\partial \hat{u}}\right)\right)^2$$

So $L_{11}L_{22} - L_{12}^2 / EG - F^2$ is "invariant".

This "tedious but routine calculation" may seem at first glance mysterious as to motivation. But it all becomes clear expressed in terms of linear algebra of quadratic forms: Think of $\begin{pmatrix} L_{11} & L_{12} \\ L_{12} & L_{22} \end{pmatrix}$ transforming under "change of basis": $\hat{L}$ as a matrix $= L$ as a matrix multiplied on one side by the Jacobian $\begin{pmatrix} \frac{\partial u}{\partial \hat{u}} & \frac{\partial u}{\partial \hat{v}} \\ \frac{\partial v}{\partial \hat{u}} & \frac{\partial v}{\partial \hat{v}} \end{pmatrix}$ and on the other by its transpose.

Then det(the $\hat{L}$ matrix) = det(Jacobian) det(Jacobian transpose) · det(L matrix)

$= (\det(\text{Jacobian}))^2 \det(L \text{ matrix})$

The proof that the Gauss curvature was both intrinsic and coordinate-choice-independent was complicated. But in practice the calculation of Gauss curvature is usually easy. In particular, if you go back to the $(u, v, f(u,v))$ example we talked about before, the calculation is very simple: Recall that we were look at the case where $f(0,0) = \vec{0}$ and $f_u(0,0) = f_v(0,0) = 0$. In this situation we found $L_{11} = -f_{uu}$, $L_{12} = -f_{uv}$, and $L_{22} = -f_{vv}$.

So the Gauss curvature at $(0,0) = f_{uu} f_{vv} - f_{uv}^2$.

In words, the Gauss curvature = the Hessian determinant of f. (This is the same item that turned up in two variable maximum/minimum problems).

So Gauss curvature $> 0 \Rightarrow$ locally the surfaces on one side of its tangent plane

Gauss curvature $< 0 \Rightarrow$ Locally the surface is partly on one side, partly on the other

and for Gauss curvature 0 the situation is not determined and has to be examined on a case-by-case basis. This is in effect the same result one had for the max/min problems:

Hessian determinant $> 0 \Rightarrow$ (local) local max or min

Hessian determinant $< 0 \Rightarrow$ "saddle point", neither local max nor local min

Hessian ~~determinant~~ $= 0$: ~~situation~~ not determined,