Scaling of Surfaces and Gauss Curvature

If $S : U^{\subset \mathbb{R}^2} \to \mathbb{R}^3$ is a surface patch and λ is a positive real number, then we can get a new surface patch by multiplying S (as a vector) by λ: namely, we define

$$S^\lambda(u,v) = \lambda \vec{S}(u,v).$$

The patch S^λ is like a magnified or shrunken version of S, depending on whether $\lambda > 1$ or $\lambda < 1$. (If $\lambda = 1$, S^λ is just S).

Gauss curvature behaves in a simple way under such "rescaling": If $(u_0, v_0) \in U$, then the Gauss curvature of S^λ at (u_0, v_0) =

$(1/\lambda^2)$ (Gauss curvature of S at (u_0, v_0)).

The proof of this is most easily carried out by looking at how things work in the case $S(u,v) = (u, v, f(u,v))$ [all patches can be moved by a rigid motion and reparam. to be like this, so this is without loss of generality] where $f_u = f_v = 0$ at $(0,0)$ and $(u_0, v_0) = (0,0)$.

Recall Gauss curvature in this case $= f_{uu} f_{vv} - f_{uv}^2$, evaluated at $(0,0)$. $S^\lambda(u,v) = (\lambda u, \lambda v, \lambda f(u,v))$. We reparam. to get the same form as $S^\lambda(\hat{u}, \hat{v}) = (\hat{u}, \hat{v}, \lambda f(\hat{u}/\lambda, \hat{v}/\lambda))$

Then Gauss curvature of S^λ at $(0,0)$ =

$F_{\hat{u}\hat{u}} F_{\hat{v}\hat{v}} - F_{\hat{u}\hat{v}}^2$ at $(0,0)$ where $F(\hat{u}, \hat{v}) = \lambda f(\hat{u}/\lambda, \hat{v}/\lambda)$.

But the Chain Rule $F_{\hat{u}\hat{u}}\big|_{(0,0)} = \lambda \cdot \frac{1}{\lambda^2} f_{uu}\big|_{(0,0)}$ and similarly for $F_{\hat{u}\hat{v}}$ & $F_{\hat{v}\hat{v}}$. So $F_{\hat{u}\hat{u}} F_{\hat{v}\hat{v}} - F_{\hat{u}\hat{v}}^2 = \frac{1}{\lambda^2}(f_{uu} f_{vv} - f_{uv}^2)$

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