Curves on Surfaces.

Let $S(u,v)$ be a (nonsingular) surface patch (i.e. $\vec{S}_u \times \vec{S}_v \neq \vec{0}$). A curve $(u(t), v(t))$ in the (u,v) domain of definition of S gives rise to a curve $S(u(t), v(t))$ "in" the surface S, i.e. the image of the curve lies in the image of S (obviously) and if S is one-to-one, all curves in S arises in this way. We are interested in the curvature behavior of such curves $\gamma(t) = S(u(t), v(t))$. As usual, we compute the tangent vector: $\gamma'(t) = \frac{du(t)}{dt}\vec{S}_u + \frac{dv(t)}{dt}\vec{S}_v$ so that $\gamma'(t)$ belongs to the tangent plane of S at $\gamma(t)$ (which is in S). We restrict our attention to arc length parameter curves, i.e. $\|\gamma'(t)\| = 1$ for all t, and we write $\gamma(s)$ to emphasize this condition. Then $T = T(s) = \frac{du(s)}{ds}\vec{S}_u + \frac{dv(s)}{ds}\vec{S}_v$ and (because $\|T\| = 1$)

$$1 = \left(\frac{du}{ds}\right)^2 E + 2F\frac{du}{ds}\frac{dv}{ds} + \left(\frac{dv}{ds}\right)^2 G$$

where $E = \langle S_u, S_u\rangle$, $F = \langle S_u, S_v\rangle$, and $G = \langle S_v, S_v\rangle$ as usual. Then a further differentiation yields

$$\frac{dT}{ds}\left(= \kappa N_\gamma\right) = \frac{d^2u}{ds^2}S_u + \frac{d^2v}{ds^2}S_v + \frac{du}{ds}\frac{d}{ds}\left(S_u\Big|_{\gamma(s)}\right) + \frac{dv}{ds}\frac{d}{ds}\left(S_v\Big|_{\gamma(s)}\right).$$

N_γ = normal of curve γ

The presence of the last two terms require a little thought. The point is that when t varies the vectors S_u and S_v, which are evaluated at $\gamma(s)$, also change. Indeed,

$$\frac{d}{ds}\left(S_u\big|_{\gamma(s)}\right) = \frac{du}{ds} S_{uu} + \frac{dv}{ds} S_{uv}$$

and
$$\frac{d}{ds}\left(S_v\big|_{\gamma(s)}\right) = \frac{du}{ds} S_{vu} + \frac{dv}{ds} S_{vv}.$$

Thus

$$\begin{aligned}\frac{dT}{ds} &= \frac{d^2u}{ds^2} S_u + \frac{d^2v}{ds} S_v \\ &\quad + \left(\frac{du}{ds}\right)^2 S_{uu} + 2\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) S_{uv} + \left(\frac{dv}{ds}\right)^2 S_{vv} \\ &= \kappa N_\gamma \quad \text{(where } \kappa, N_\gamma \text{ are as usual for space curves, } N_\gamma = \text{curve normal of } \gamma\text{)}.\end{aligned}$$

There is of course no reason why the curve normal N_γ has to be the surface normal N (Example: think about a nonequatorial "parallel of latitude on the sphere). It is of interest to look, however, a the "normal component" of $\frac{dT}{ds}$ here: namely, $\langle N, \frac{dT}{ds}\rangle$.
This satisfies

$$-\langle N, \frac{dT}{ds}\rangle = \left(\frac{du}{ds}\right)^2 L_{11} + 2\left(\frac{du}{ds}\right)\left(\frac{dv}{ds}\right) L_{12} + \left(\frac{dv}{ds}\right) L_{22}.$$

(the $-$ sign arises from $L_{11} = -\langle S_{uu}, N\rangle$).

The tangent part of $\frac{dT}{ds}$ is given by

$$\frac{d^2u}{ds^2} S_u + \frac{dv^2}{ds^2} S_v + \left(\frac{du}{ds}\right)^2 T(S_{uu}) + 2\left(\frac{du}{ds}\frac{dv}{ds}\right) T(S_{uv}) + \left(\frac{dv}{ds}\right)^2 T(S_{vv})$$

where $T(\ )$ = tangent part as earlier.

The normal part of the $\frac{dT}{ds}$ vector is determined entirely by T itself. That is, by $\frac{du}{ds}$ & $\frac{dv}{ds}$. But the tangent part depends also on the second derivatives of γ, $\frac{d^2u}{ds^2}$ and $\frac{d^2v}{ds^2}$.

Note that for a given tangent vector (i.e. given $\frac{du}{dt}, \frac{dv}{dt}$), the minimum possible value for $\left\|\frac{dT}{ds}\right\|$ is obtained if the tangent part of $\frac{dT}{ds}$ is 0, in which case the value of $\left\|\frac{dT}{ds}\right\|$ is

$$\left| \left(\frac{du}{ds}\right)^2 L_{11} + 2\frac{du}{ds}\frac{dv}{ds} L_{12} + \left(\frac{dv}{ds}\right)^2 L_{22} \right|.$$

(We could alway realize this minimum, too, by taking γ to be a "section" of S by the plane contain N and the specified tangent vector).

We set K_g = the length of the tangent (to S) part of $\frac{dT}{ds}$ and $|K_n|$ (n for normal) = the length of the normal part.
Then the curvature K of γ as a space curve satisfies

$$K^2 = K_g^2 + K_n^2$$

Moreover, $K|\cos\theta| = |K_n|$
where θ = the angle between the surface normal N and the curve normal N_γ.
The seemingly strange notation $|K_n|$ suggests that there is a K_n with a $\pm$ sign: this is just $K_n = \langle(\text{normal part of } \frac{dT}{ds}), N\rangle$
where N = the surface normal! $-K_n = \left(\frac{du}{dt}\right)^2 L_{11} + 2\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) L_{12} + \left(\frac{dv}{dt}\right)^2 L_{22}$.
The standard theory of symmetric quadratic forms gives that the Gauss curvature at the point = the product of the maximum and minimum values of the "normal curvature" K_n of arclength parameter curves through the point.

Recalling that the tangent parts of S_{uu}, S_{uv}, and S_{vv} are intrinsic, we see also that the tangent part of dT/ds is also intrinsic.

Definition: A curve $\gamma(s)$ (with arclength parameter) is an arclength parameter geodesic in S if the tangent part of $\frac{dT}{ds}$ is 0 for all s values (for which the curve is defined).

It is not clear a priori that geodesics exist! All we have is an equation that they have to satisfy everywhere, i.e., for all s values.

Let us write this equation a little more neatly.

First write
$$
\begin{aligned}
T(S_{uu}) &= \Gamma_{11}^{1} S_u + \Gamma_{11}^{2} S_v \\
T(S_{uv}) &= \Gamma_{12}^{1} S_u + \Gamma_{12}^{2} S_v \\
T(S_{vu}) &= \Gamma_{21}^{1} S_u + \Gamma_{21}^{2} S_v \\
T(S_{vv}) &= \Gamma_{22}^{1} S_u + \Gamma_{22}^{2} S_v .
\end{aligned}
$$

Since $S_{uv} = S_{vu}$, we see that $\Gamma_{12}^{1} = \Gamma_{21}^{1}$ and $\Gamma_{12}^{2} = \Gamma_{21}^{2}$. The Γ's are function on the surface: they can be and usually are different at different points of the patch S.

In this notation, separating S_u and S_v components:

$$T\left(\frac{dT}{d\sigma}\right) = \left(\frac{d^2u}{ds^2} + \left(\frac{du}{ds}\right)^2\Gamma^1_{11} + 2\frac{du}{ds}\frac{dv}{ds}\Gamma^1_{12} + \left(\frac{dv}{ds}\right)^2\Gamma^1_{22}\right)S_u$$
$$+\left(\frac{d^2v}{ds^2} + \left(\frac{du}{ds}\right)^2\Gamma^2_{11} + 2\frac{du}{ds}\frac{dv}{ds}\Gamma^2_{12} + \left(\frac{dv}{ds}\right)^2\Gamma^2_{22}\right)S_v.$$

In particular, a curve $\gamma(s) = (u(s), v(s))$ is a geodesic if and only if the $u(s)$ and $v(s)$ satisfy two ("coupled") ordinary differential equations

$$0 = \frac{d^2u}{ds^2} + \left(\frac{du}{ds}\right)^2\Gamma^1_{11}\Big|_{(u(s),v(s))} + 2\frac{du}{ds}\frac{dv}{ds}\Gamma^1_{12} + \left(\frac{dv}{ds}\right)^2\Gamma^1_{22}$$

$$0 = \frac{d^2v}{ds^2} + \left(\frac{du}{ds}\right)^2\Gamma^2_{11} + 2\frac{du}{ds}\frac{dv}{ds}\Gamma^2_{12} + \left(\frac{dv}{ds}\right)^2\Gamma^2_{22}$$

This looks nicer if we start writing u_1, u_2 for u and v. Namely, the equations become

$$0 = \frac{d^2u_1}{ds^2} + \sum_{i,j}\Gamma^1_{ij}\frac{du_i}{ds}\frac{du_j}{ds}$$

$$0 = \frac{d^2u_2}{ds^2} + \sum_{i,j}\Gamma^2_{ij}\frac{du_i}{ds}\frac{du_j}{ds}.$$

It is worth noting that we did not really make much use of arclength parameter here: it was only relevant in talking about the curvature of the curve as a space curve and so on. In particular, we could talk about geodesics as being curve that satisfied these equations whether or not s was arclength parameter. It turns out this does not matter too much: a geodesic in this extended sense always has $\left\| \frac{dT}{ds} \right\|$ constant, though not necessarily 1. But it is convenient to allow this great freedom, so from now on we call any curve γ satisfying the two equations a geodesic and call it an "arclength parameter geodesic" explicitly if it does have arclength parameter.

It is not hard to see why geodesics in this more general sense are parameterized "proportional to arc length": For any curve $\gamma(t)$ at all (in a surface S), $\frac{d}{dt} < \gamma', \gamma' >$ $= 2 < \gamma'', \gamma' > = 2 <$ tangent part of $\gamma'', \gamma' >$ since γ' is tangent to S (lies in the tangent plane) [Here $'$ = t-derivative]. So if tangent part of γ'' (= tangent part of $\frac{d^2\gamma}{dt^2}$) $= 0$, then $< \gamma', \gamma' >$ is constant, which is the same as $\left\| \frac{d\gamma}{dt} \right\|$ is constant, which is itself the same as proportional-to-arclength parameterization.

Geodesics (with arclength parameter) are important because they are the analogues for surfaces in general of straight lines in the plane or great circles on the sphere. Namely, if $\gamma(s)$ is an (arclength parameter) geodesic then for a fixed s_0 and all s close enough to s_0, the length of any curve σ in S connecting $\gamma(s_0)$ to $\gamma(s)$ is at least $|s-s_0|$, with equality if and only if σ is γ up to reparametrization. In summary form, γ is locally the unique shortest connection between points on it. (Think about great circles to see why we need "locally" and cannot just say that γ is the shortest connection).

Proving this will take a little work! but intuitively, it seems likely: a geodesic does not "turn" in terms of the tangential component of its acceleration, so one expects it is heading straight for where it goes.