Basic Information on Quadratic Forms in Two Dimensions.

We need to understand quadratic functions on $\mathbb{R}^2$, that is, functions of the form $Q((x,y)) = ax^2 + 2bxy + cy^2$ $x, y \in \mathbb{R}$. It is convenient for this purpose to think of Q as arising from the matrix $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ as follows: $Q((x,y)) = < A\begin{pmatrix} x \\ y \end{pmatrix}, (x,y)>$ where $A\begin{pmatrix} x \\ y \end{pmatrix} =$ the "matrix product" of $\begin{pmatrix} a & b \\ b & c \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ $= (ax+by, bx+cy)$. In particular, we are going to analyze Q in terms of the "eigenvalues" of the matrix A.

Recall that an eigenvalue λ of a matrix A is a solution of $\det(A - \lambda I) = 0$. In our case, this becomes $\det\begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix} = 0$. This quadratic equation for λ has real roots: the equation written out is $(a-\lambda)(c-\lambda) - b^2 = 0$ or $\lambda^2 - (a+c)\lambda + ac - b^2 = 0$. Its solutions are $\lambda = \frac{1}{2}(a+c \pm \sqrt{(a+c)^2 - 4(ac-b^2)}$ $= \frac{1}{2}(a+c \pm \sqrt{(a-c)^2 + 4b^2}$. Since $(a-c)^2 + 4b^2 \geq 0$, these solutions are real numbers (in general, eigenvalues can be complex numbers).

There are two cases:

(1) $(a-c)^2 + 4b^2 > 0$ and (2) $(a-c)^2 + 4b^2 = 0$.

In case (2), $a = c$ and $b = 0$. So $Q((x,y)) = a(x^2+y^2)$. ($A = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$). The function Q is constant on the unit circle $\{(x,y) : x^2 + y^2 = 1\}$, with cons. value a.

Case (1), $(a-c)^2+4b^2>0$, is more interesting but less easy. In this case there are two real eigenvalues λ_1, λ_2 with, say, $\lambda_1>\lambda_2$ namely $\lambda_1=\frac{1}{2}\left(a+c+\sqrt{(a-c)^2+4b^2}\right)$ and $\lambda_2=\frac{1}{2}\left(a+c-\sqrt{(a-c)^2+4b^2}\right)$.

Now $\det\begin{pmatrix} a-\lambda & b \\ b & c-\lambda \end{pmatrix}=0$ if and only if there is a nonzero vector (x,y) with $A\begin{pmatrix} x \\ y \end{pmatrix}=\lambda(x,y)$, i.e. $ax+by=\lambda x$ & $bx+cy=\lambda y$.

[This is just a restatement of the basic idea of linear algebra that a system with two equations and two unknowns that is homogeneous has a nontrivial solution if and only if it has determinant $=0$].

Let $\vec{v}_1$ be an "eigenvector" $(\neq\vec{0})$ for λ_1, $\vec{v}_2$ for λ_2, i.e. $A\vec{v}_1=\lambda_1\vec{v}_1$ and $A\vec{v}_2=\lambda_2\vec{v}_2$.

Without loss of generality we can take $\|v_1\|=1$ & $\|v_2\|=1$.

Lemma: If $\lambda_1\neq\lambda_2$, then $\langle\vec{v}_1,\vec{v}_2\rangle=0$.

Proof: Direct calculation shows that $\langle A\vec{v}_1,\vec{v}_2\rangle=\langle\vec{v}_1,A\vec{v}_2\rangle$ for any two vectors $\vec{v}_1,\vec{v}_2$: this is just because A is symmetric.

But $\langle A\vec{v}_1,\vec{v}_2\rangle=\langle\lambda_1\vec{v}_1,\vec{v}_2\rangle=\lambda_1\langle\vec{v}_1,\vec{v}_2\rangle$ while

(with v_1, v_2 eigenvectors) $\langle\vec{v}_1,A\vec{v}_2\rangle=\langle\vec{v}_1,\lambda_2\vec{v}_2\rangle=\lambda_2\langle\vec{v}_1,\vec{v}_2\rangle$

So $\langle A\vec{v}_1,\vec{v}_2\rangle=\langle\vec{v}_1,A\vec{v}_2\rangle$ gives $\lambda_1\langle\vec{v}_1,\vec{v}_2\rangle=\lambda_2\langle\vec{v}_1,\vec{v}_2\rangle$

or $(\lambda_1-\lambda_2)\langle\vec{v}_1,\vec{v}_2\rangle=0$. Since $\lambda_1-\lambda_2\neq0$, it follows that $\langle\vec{v}_1,\vec{v}_2\rangle=0$.

Now suppose (without loss of generality) that, as before, $\lambda_1 > \lambda_2$. If $\vec{u}$ is a unit-length vector then $\vec{u} = \alpha \vec{v_1} + \beta \vec{v_2}$ where $\alpha^2 + \beta^2 = 1$ (v_1, v_2 are unit and perpendicular and are hence an orthonormal basis for $\mathbb{R}^2$). Then

$$\langle A\vec{u}, \vec{u} \rangle = \langle \alpha\lambda_1 v_1 + \beta\lambda_2 v_2, \alpha v_1 + \beta v_2 \rangle$$
$$= \alpha^2 \lambda_1 + \beta^2 \lambda_2$$

(since $\langle v_1, v_2 \rangle = 0$ while $\langle v_1, v_1 \rangle = 1$ & $\langle v_2, v_2 \rangle = 1$

Clearly $\alpha^2 \lambda_1 + \beta^2 \lambda_2$ is as large as possible (with α, β satisfying $\alpha^2 + \beta^2 = 1$) when $\alpha = \pm 1$ & $\beta = 0$ while it is as small as possible when $\alpha = 0$ & $\beta = \pm 1$. Thus, in summary:

The eigenvalues of $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$ are the maximum and minimum values of $ax^2 + 2bxy + cy^2$ on the unit circle. If $ax^2 + 2bxy + cy^2$ is not constant on the unit circle, then the maximum occurs exactly twice, at two antipodal points $\pm v_1$, and the minimum exactly twice at $\pm v_2$ for some v_2, and $\langle v_1, v_2 \rangle = 0$. The vectors $\pm v_1$ and $\pm v_2$ are the (unit) eigenvectors corresponding to the eigenvalues $\lambda_1 = \max$ and $\lambda_2 = \min$ of $ax^2 + 2bxy + cy^2$ on the unit circle.

The product $\lambda_1 \lambda_2$ of the eigenvalues of $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$

Proof of the Lemma: λ_1 and λ_2 are the solutions of $(a-\lambda)(c-\lambda)-b^2=0$. By usual algebra, their product is the constant term of the equation, namely $ac-b^2$. $\square$

We are going to combine this lemma with the observations about λ_1 and λ_2 as maximum values and minimum value to prove the following comparison (we need the lemma for the second statement here):

Theorem: Suppose $A=\begin{pmatrix} a & b \\ b & c\end{pmatrix}$ and $\hat{A}=\begin{pmatrix} \hat{a} & \hat{b} \\ \hat{b} & \hat{c}\end{pmatrix}$ are two symmetric matrices. Let $Q(x,y)=ax^2+2bxy+cy^2$ and $\hat{Q}(x,y)=\hat{a}x^2+2\hat{b}xy+\hat{c}y^2$. And suppose that for all (x,y) with $x^2+y^2=1$, $Q(x,y)\geq \hat{Q}(x,y)$. Then

(1) If λ_1,λ_2 are the max and min eigenvalues of A and $\hat{\lambda}_1,\hat{\lambda}_2$ are the max and min eigenvalues of $\hat{A}$, then $\lambda_1\geq\hat{\lambda}_1$ and $\lambda_2\geq\hat{\lambda}_2$

(2) If $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are nonnegative, then $\det A = ac-b^2$ is $\geq$ $\det\hat{A}=\hat{a}\hat{c}-\hat{b}^2$.

Proof: Statement (2) follows from $\det A=\lambda_1\lambda_2$ and $\det\hat{A}=\hat{\lambda}_1\hat{\lambda}_2$ together with statement (1): If $\lambda_1\geq\hat{\lambda}_1\geq 0$ and $\lambda_2\geq\hat{\lambda}_2\geq 0$, then $\lambda_1\lambda_2\geq\hat{\lambda}_1\hat{\lambda}_2$.

To prove statement (1), suppose $\hat{v}_1$, in unit circle is such that $\hat{Q}(\hat{v}_1)$ is maximal, so that $\hat{Q}(\hat{v}_1)=\hat{\lambda}_1$. Then $\hat{\lambda}_1=\hat{Q}(\hat{v})$ $\leq Q(\hat{v}_1)\leq$ max Q on unit circle $=\lambda_1$. For the $\lambda_2,\hat{\lambda}_2$ estimate, suppose v_2 ($\in$ unit circle) is such that

$Q(v_2)$ is minimal (for Q on unit circle) and hence $\lambda_2 = Q(v_2)$. Then $\lambda_2 = Q(v_2) \geq \hat{Q}(v_2) \geq \min \hat{Q}$ on unit circle $= \hat{\lambda}_2$. $\square$

Application: Suppose $f(x,y)$ and $\hat{f}(x,y)$ are such that $f(0,0) = 0 = \hat{f}(0,0)$ and $f_x|_{(0,0)} = f_y|_{(0,0)} = 0$ and $\hat{f}_x|_{(0,0)} = \hat{f}_y|_{(0,0)} = 0$. And finally suppose $f(x,y) \geq \hat{f}(x,y)$ for all (x,y) in a neighborhood of $(0,0)$. Then, supposing that $\hat{f}(x,y) \geq 0$ for all (x,y) in a neighborhood of $(0,0)$, the Gauss curvature of the graph $z = f(x,y)$ at $(0,0)$ is $\geq$ the Gauss curvature of $z = \hat{f}(x,y)$ at $(0,0)$.

Proof of the application: Set $Q(x,y) = f_{xx}|_{(0,0)} x^2 + 2 f_{xy}|_{(0,0)} xy + f_{yy}|_{(0,0)} y^2$ and similarly for $\hat{Q}$ (with f replaced by $\hat{f}$). Note that $Q(x,y) = \frac{d^2}{dt^2} f(tx,ty)|_{(0,0)}$ [This is a standard step in the development of Taylor's expansion in two variables. It is proved via the Chain Rule: $\frac{d}{dt} f(tx,ty) = x f_x|_{(tx,ty)} + y f_y|_{(tx,ty)}$

So $\frac{d^2}{dt^2} f(tx,ty)|_{(0,0)} = x(x f_{xx}|_{(0,0)} + y f_{xy}|_{(0,0)})$

$+ y(x f_{xy}|_{(0,0)} + y^2 f_{yy}|_{(0,0)})$

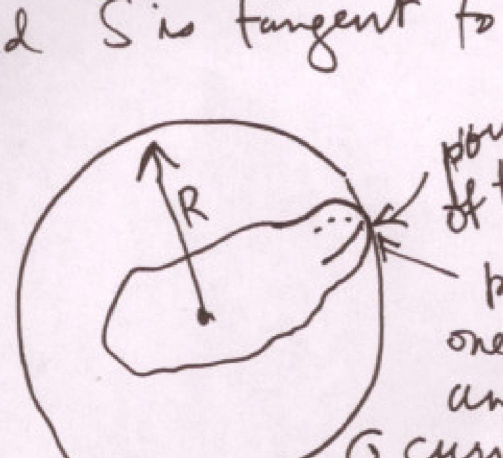
$= Q(x,y)$]. We fix $(x,y) \in$ unit circle, i.e. $x^2+y^2=1$.

(Supposing still $(x,y) \in$ unit circle):

Now $f(tx,ty)$ and $\hat{f}(tx,ty)$ are both $=0$ when $t=0$ and have first t-derivative $=0$ when $t=0$ (since $f_x = f_y = 0$ at $(0,0)$). So $f(tx,ty) \geq \hat{f}(tx,ty)$ for small $|t|$ implies $(d^2/dt^2)\, f(tx,ty)|_{t=0} \geq \frac{d^2}{dt^2} \hat{f}(tx,ty)|_{t=0}$. Thus $Q(x,y) \geq \hat{Q}(x,y)$, from $Q(x,y) = \frac{d^2}{dt^2} f(tx,ty)|_{t=0}$ and same for $\hat{Q}$.

The conclusion of the Application follows from statement (2) of the theorem, together with the fact that $\hat{\lambda}_1$ and $\hat{\lambda}_2$ are nonnegative, this latter following from $\hat{f} \geq 0$ near $(0,0)$. □

This application completes the proof already given in class that a closed surface S in $\mathbb{R}^3$ has a point of positive Gauss curvature: Choose an arbitrary $p \in \mathbb{R}^3$ say $p = (0,0,0)$. The function $x^2+y^2+z^2$ has a positive maximum on S, say R^2 $(R>0)$. Then the sphere of radius R around $(0,0,0)$ contains S and S is tangent to the sphere at some point.



point of tangency. Graphing the S surface and the sphere over the tangent plane at the point (of both of them) puts one in the position of the Application and gives

G. curvature of $S \geq$ Gauss curve of sphere $1/R^2 > 0$.