Homework IV: Due October 24, 2008

1. Set $\gamma_t(s) = ((1+t)\cos s, (1-t)\sin s)$

(a) Argue geometrically (thinking of scaling horizontally and vertically) that

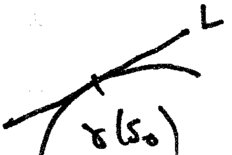
$$\left.\frac{dA(t)}{dt}\right|_{t=0} = 0.$$

(b) Compute (in our standard notation) the functions $a(s)$ and $b(s)$ in this case and verify the formula

$$\left.\frac{dA(t)}{dt}\right|_{t=0} = -\int b(s)\,ds$$

2. Suppose that $\gamma(s)$ is a plane curve with arc-length parameter. And suppose that the plane curve curvature of γ is nonzero at $s=s_0$. Prove that, if L is the line tangent to γ at $s=s_0$, then $L \cap \gamma$ is a single point "near s_0". Precisely, prove $\exists\ \varepsilon > 0$ such that

$$L \cap \gamma([s_0-\varepsilon, s_0+\varepsilon]) \text{ is a single point.}$$



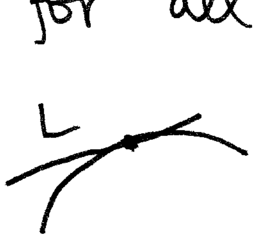
[Suggestion: If N_0 = the normal to γ at $s=s_0$, then N_0 is normal to L. What are the first and second derivatives of $\langle N_0, \gamma(s)\rangle$?]

3. Suppose $\gamma : [0, L] \to \mathbb{R}^2$ is a simple closed curve that is smoothly closed (i.e. $\gamma(0) = \gamma(L)$, $\gamma'(0) = \gamma'(L)$, $\gamma''(0) = \gamma''(L)$, etc.). As usual, we suppose γ has arc-length parameter. Suppose also that γ is traversed counterclockwise.

(a) Show that there is an $s_0 \in [0, L]$ such that the plane curve curvature of γ at $s = s_0$ is positive and the (plane curve) normal of γ at $s = s_0$ points into the inside of γ. [Suggestion: Choose $p_0 \in$ inside of γ and choose s_0 such that distance from p_0 to $\gamma(s)$ is maximal at $s = s_0$].

Now assume plane curve curvature > 0 for all s.

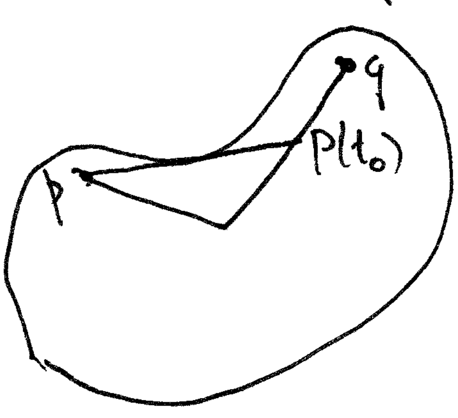
(b) Show that the tangent line L of γ lies outside γ (or on γ) near $\gamma(s)$ for all $s \in [0, L]$



4. Use problem 3 to show that if $\gamma : [0, L] \to \mathbb{R}^2$ is a counter-clockwise traversed, simple, smoothly closed, arclength parameter curve in the plane with plane curve curvature nowhere $= 0$, then the interior of γ (inside of γ) is a convex set.

[A set A in $\mathbb{R}^2$ is <u>convex</u> if for all $p, q \in A$, the line segment from p to q $\subset A$].

The following outline is suggested: If $p, q \in$ interior of γ, then there is a polygonal arc $P(t)$ from p to q — you may assume this without proof. Consider then the smallest $t = t_0$ such that the line segment from p to $P(t_0)$ intersects γ, if such a t exists. Look at the picture:



Show this cannot happen so that the segment from p to $P(t)$ is in Interior(γ) for all t, in particular for q.