Summary of Surface Theory (things to know for Midterm II)

Definition of surface patch $S(u,v)$ $(S_u \times S_v \neq 0)$
Unit normal $S_u \times S_v / \|S_u \times S_v\|$
Definition of E, F, G, $L_{11} (= L)$, $L_{12} (= M)$, $L_{22} (= N)$
Definition of Gauss curvature $(= (L_{11}L_{22} - L_{12}^2)/(EG - F^2))$
Gauss curvature independent of reparametrization (calculation: see handout)
Gauss curvature = determinant of "second fundamental form" (= second fundamental form determinant evaluated on orthonormal basis v_1, v_2, i.e.,
$\mathrm{II}(v_1,v_1)\,\mathrm{II}(v_2,v_2) - (\mathrm{II}(v_1,v_2))^2$: independent of choice of orthonormal basis — proof by rotation calculation and $= (L_{11}L_{22} - L_{12}^2)/(EG - F^2)$: calculation by Gram-Schmidt process applied to S_u, S_v and evaluation: see handout)
Mean curvature $\mathrm{II}(v_1,v_1) + \mathrm{II}(v_2,v_2) =$
$(EN + GL - 2FM)/(EG - F^2)$ (know calculation)

Basic example: $S(u,v) = (u, v, f(u,v))$
with $f_u = f_v = 0$ at $(u,v) = (0,0)$
Gauss curvature $= f_{uu}f_{vv} - f_{uv}^2$
Mean curvature $= f_{uu} + f_{vv}$

Calculate for sphere radius R.
Gauss $= 1/R^2$
mean $= 2/R$

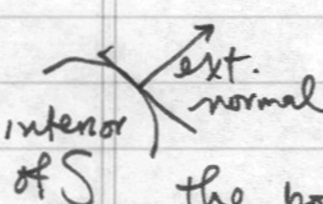
Fundamental observations: (1) Gauss curvature > 0 (at pt.)
$\Rightarrow$ surface is on one side of its tangent plane locally (in neighborhood of point)
(2) $g \geq f$, $g(0,0) = f(0,0)$ then Gauss curvature of $(u,v,g(u,v)) \geq$ G. curv. of $(u,v,f(u,v))$ at $(0,0)$

Note that proof of point (2) uses: determinant of quadratic form = product of max and min values on unit circle: see quadratic form handout.

Consequence of point (2): If S is a closed bounded surface (no edges), then $\exists\, p \in S$ such that Gauss curvature at $p > 0$.

Proof (outline): Choose $p \in S$ so that $\|p\|$ is maximal among $\|\;\|$ of point of S (p is "farthest point" from $\vec{0}$). Then S touches sphere around $\vec{0}$ of radius $\|p\|$ at the point p but lies inside sphere. Use point (2) after graphing over tangent plane.

Corollary: If closed bounded surface S has $K>0$ everywhere then for each point $q \in S$, the interior of S lies locally on the opposite side of the tangent plane of S at q from the exterior normal. [We assume without proof here that S divides $\mathbb{R}^3$ into two components.]



Proof of Corollary: Clearly true for $q =$ the point p in the proof that $\exists$ point of positive curvature. Then true everywhere on S by continuity.

Corollary $\Rightarrow$ ($K>0$ everywhere then S bounds convex set: cf. homework problem sequence)

Big theorem (Gauss's Theorema Egregium):
Gauss curvature is intrinsic (depends only on E F G and their derivatives).

KNOW PROOF! in detail, including all formulas

Proof steps:

(1) $\langle S_{uu}, S_u\rangle$, $\langle S_{uv}, S_u\rangle$, $\langle S_{uu}, S_v\rangle$, etc. are intrinsic.

Reason: $\langle S_{uu}, S_u\rangle = \frac{1}{2}\frac{\partial}{\partial u}\langle S_u, S_u\rangle = \frac{1}{2}E_u$

$$\langle S_{uu}, S_v\rangle = \frac{\partial}{\partial u}\langle S_u, S_v\rangle - \langle S_u, S_{uv}\rangle$$

$$= \frac{\partial}{\partial u}F - \frac{1}{2}\frac{\partial}{\partial v}E, \text{ etc.}$$

(2) $T(S_{uu})$, $T(S_{uv})$, $T(S_{vv})$ (T = tangent part) are intrinsic:

Reason: If $T(S_{uu}) = aS_u + bS_v$ then

$\langle T(S_{uu}), S_u\rangle = \langle S_{uu}, S_u\rangle = aE + bF$

$\langle T(S_{uu}), S_v\rangle = \langle S_{uu}, S_v\rangle = aF + bG$

So $aE + bF$ = intrinsic quantity

$aF + bG$ = intrinsic quantity

so can solve for a, b (since $\det\begin{pmatrix} E & F \\ F & G\end{pmatrix} \neq 0$) intrinsically.

(3) Calculate $\langle (S_{uu})_v, S_v\rangle$ and (same quantity) $\langle (S_{uv})_u, S_v\rangle$ by

$$\langle (S_{uu})_v, S_v\rangle = \frac{\partial}{\partial v}\langle S_{uu}, S_v\rangle - \langle S_{uu}, S_{vv}\rangle$$

$$= \frac{\partial}{\partial v}\langle S_{uu}, S_v\rangle - \langle T(S_{uu}), T(S_{vv})\rangle - \langle \overset{L_{11}}{N(S_{uu})}, \overset{L_{22}}{N(S_{vv})}\rangle$$

and similarly for $\langle (S_{uv})_u, S_v\rangle$, set equal to get

$L_{11}L_{22} - L_{12}^2$ = intrinsic quantity.

Example: (u, v, f_{uv}) ← in this case, at $(0,0)$
$T(S_{uu}) = 0$ (at $(0,0)$) so formula simplies
Know details of this! Work this out!
Use this to check again: Gauss curvature of
sphere radius 1 $= +1$

Ideas related to Gauss Bonnet Theorem:

Area integration $\int f \, dA \overset{def}{=} \int f \sqrt{EG - F^2} \, du dv$

Proof that this is independent of parameterization:
By Lagrange identity, $\sqrt{EG - F^2} = \| S_u \times S_v \|$
and $\|S_u \times S_v\|$ changes by | Jacobian determinant |
under parameterization: compare with change of
variables for double integrals formula
[Note: This can be done for "abstract surface" by
noting that $EG - F^2$ changes by $|\text{Jacobian}|^2$:
this was already calculated in the handout
verifying that Gauss curvature was parametrization
independent and "intrinsic": see that handout)

Gauss Bonnet Theorem (first statement)
If S is a closed, bounded surface (no edges)
then

$$\int_S K \, dA$$

depends only on the topology of S.

Exact meaning: Two surfaces S_1 and S_2 are <u>homeomorphic</u> (or "have the same topology") if there is a continuous, 1-1, onto function $F: S_1 \to S_2$ with F^{-1} continuous. (In this particular case, continuity of F^{-1} is automatic but we include it as part of the definition for convenience).

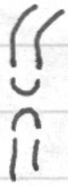
Corollary: $$\int_S K \, dA = 2\pi(2-2g)$$

for a "g-hole" surface (surface of genus g).

Proof of Corollary: True for $g=0$ (sphere).

Inductively: Cut along curve shown (assuming wolog that piece of cylinder is as shown) then

cylinder



adding spherical caps (smoothed at edges) as shown adds 4π to curvature integral but then we have $g-1$ hole surface. So $4\pi + \int_{\text{genus } g} K\,dA = \int_{\text{genus } g-1} K\,dA$. $\square$

Formula not proved
proof ↓ not required!

Outline proof of Gauss Bonnet: Local formula

$$\int_{\text{triangle}} K\,dA + \pi + \text{edge terms} = \sum \text{angles of triangle}.$$

where edge terms cancel out  (for this we assume S is orientable: for abstract surface case, maybe nonorientable, we have to use another argument: <u>not</u> presented nor required)

Adding up gives

no. of vertices in triangulation

$$2\pi(V) = \int K\,dA + \pi \cdot F$$

no. of faces

$$\int K\,dA = 2\pi\left(V - \tfrac{1}{2}F\right).$$ Since $2E = 3F$ (each edge is in two faces),

$$\int K\,dA = 2\pi(V - E + F).$$

The number $V-E+F$ is called the Euler characteristic for that triangulation. But note that since $\int K\,dA$ is independent of the triangulation, $V-E+F$ is the same! for every triangulation. It is called the Euler characteristic of S.

Note that we can show inductively that $\chi(\text{genus } g \text{ surface}) = 2-2g$.

$g=0$ works.

Inductive step: Wolog cutting circle = union of edges of triangulation.

$-E+V=0$ for circle so "doubling" circle makes no change. Capping adds 1 to χ sum. So

$$\chi(g \text{ hole}) + 2 = \chi(g-1 \text{ hole})$$

or $$\chi(\text{genus } g) = \chi(\text{genus } g-1) - 2. \quad \square$$

From homework:
Gauss map $\Gamma : S \to$ unit sphere is one-to-one and onto if S is a closed surface (bounded, no edges) with $K > 0$ everywhere.
Calculation (using $(u, v, f(u,v))$ form) shows |Jacobian determinant of Gauss map| = $|K|$ ($= K$ in this case).
Change of variables for double integrals then show

$$\int_S K \, dA = \text{area of unit sphere} = 4\pi.$$

Thus we have complete proof that

$$\int_S K \, dA = 2\pi \, \chi(S) \quad \text{for}$$

a closed bounded surface in $\mathbb{R}^3$ with Gauss curvature $K > 0$ everywhere.
(Note that S is homeomorphic to a sphere, hence genus $= 0$, because Γ is a homeomorphism, in fact, differentiable with a differentiable inverse — differentiability of the inverse follows from Jacobian determinant $\neq 0$ & Inverse Function Theorem)