Sample Problems for Midterm II:

1. (a) Write down and prove the intrinsic formula for $\langle S_{uu}, S_v \rangle$

   (b) Verify that this formula works for $(u, v, f(u,v)) = S(u,v)$ (without assuming $f_u = f_v = 0$ at the point: verify for general point!)

2. (a) Write down the Gauss curvature formula derived from $\langle (S_{uu})_v, S_v \rangle = \langle (S_{uv})_u, S_v \rangle$ assuming $T(S_{\cdot\cdot}) = 0$ at the point.

   (b) Check that for $(u, v, f(u,v))$ with $f_u = f_v = 0$ at the point, the formula from part (a) actually gives the expected $f_{uu} f_{vv} - f_{uv}^2$

3. (a) Prove, by writing the surface in the form $(u, v, f(u,v))$ ($f_u = f_v = 0$ at $(u,v) = (0,0)$), that the Gauss curvature of the sphere of radius R is $1/R^2$.

   (b) Discuss how part (a) is consistent with the Gauss-Bonnet Theorem

4. Prove that every closed bounded surface in $\mathbb{R}^3$ has point of positive Gauss curvature.

5. Is it possible for a no-edge surface in $\mathbb{R}^3$ to have negative Gauss curvature everywhere if it is unbounded? (Suggestion: Use problem 6)

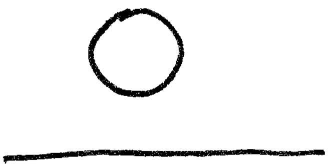
6. Let $S(u,v) = (f(u), g(u)\cos v, g(u)\sin v)$

   $(f')^2 + (g')^2 \equiv 1$, $f'(u) > 0$ $g(u) > 0$ ($'$ = u derivative)

   Show that the Gauss curvature of S is $-g''(u)/g(u)$ $''$ = 2nd u derivative

   (Suggestion: Wolog, you can look at points with $v = 0$ only)

7. Discuss how problem 6 can be used to verify that $\int K \, dA = 0$ for the "torus of revolution" obtained by rotating a circle around an axis as in the picture



u axis

8. Let $S(u,v) = (x(u), y(u), v)$ where $(x')^2 + (y')^2 = 1$. Show that the Gauss curvature of $S = 0$ everywhere.
(Suggestion: Recall that Gauss curvature depends only on E, F, and G)

9. Suppose S is a closed bounded (no edges) surface in $\mathbb{R}^3$ and that p is a point where the z-coordinate is maximal (among points on S). Show the Gauss curvature of S at $p \geq 0$. Can it be that Gauss curvature at $p = 0$?

10. Suppose S is a closed bounded surface in $\mathbb{R}^3$ the interior of which contains an open ball of radius R. Show that there is a point $p \in S$ with Gauss curvature $\leq 1/R^2$
(Suggestion: Recall that if G. curv. $\geq 1/R^2$ everywhere on S then the interior of S is convex. Keep the center of the ball fixed and increase the radius until "first contact".)