Surfaces with $K>0$ Everywhere: Solutions of the Sequence of Homework Problems.

Important stuff

Set-up: S is a closed surface (bounded, no edge) in $\mathbb{R}^3$ with Gauss curvature positive at each point.

Step 1: Show that a surface with $K>0$ lies locally on one side of its tangent plane.

Proof: Wolog, surface has form (locally) $(u,v,f(u,v))$ with $f_u=f_v=0$ at $(u,v)=(0,0)$. In this situation, $K|_{(0,0)} = f_{uu}f_{vv}-f_{uv}^2$. By standard calculus, $K|_{(0,0)}>0 \Rightarrow f$ has a strict local max or min at $(0,0) \Rightarrow f$ lies on one side of tangent plane $\blacksquare$

Note that f lies (in this case) on same side as direction of normal $\Leftrightarrow f_{uu}>0 \Leftrightarrow$ mean curvature $H<0$ (Mean curvature sign convention: sphere has + m.curv. relative to exterior unit normal).

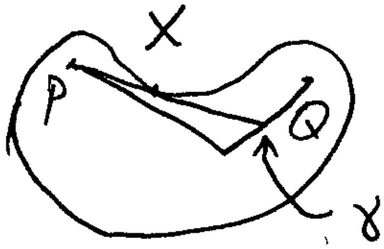
Step 2: At every point, S lies locally on the side of its tangent plane opposite the exterior normal.

Proof: According to the Note to Step 1, we need only show $H>0$ everywhere, H calculated relative to exterior unit normal. Now H is never zero since $K>0 \Rightarrow$ eigenvalues of second fundamental form are same sign & neither is 0. (we use here $K=$ product of eigenvalues). So H cannot change sign. So it is enough to find one point of S where $H>0$. This point is provided by the proof that every closed surface has a point with $K>0$. Namely in our case choose $P_0 \in \mathbb{R}^3$ and look at the point $p \in S$ with $\|p - P_0\|$ maximal among $\|\cdot - P_0\|$, $\cdot$ = point of S. Then $N(p)$ = exterior normal of S = exterior normal of sphere around P_0 through p. Using eigenvalues of second fund form of S = max, min values on unit circle, we get (as in $K>0$ at p proof) that H at p (H for S)

(ext normal used) $\geq$ mean curvature of sphere (rel. to exterior normal) >0, as required. Alternatively, you could use the "Note" at end of Step 1 plus fact that S is on opposite side of tangent plane from exterior normal N at p. [This is easier!]

Step 3: The interior of S is convex.

Proof: Suppose $P, Q \in S$. Then (as we are assuming from topology), there is a polygonal curve $\gamma : [0,1] \to$ interior of S with $\gamma(0) = P$ and $\gamma(1) = Q$. (Wolog γ has proportional to arclength parameter). We want to show that the straight line from P to Q is $\subset$ interior of S. For contradiction, suppose not. Then set t_0 = minimum t such that the line from P to $\gamma(t)$ interesects S. (Such t exists since $\{t : \text{line from } P \text{ to } \gamma(t) \cap S \neq \emptyset\}$ is closed, so t_0 = greatest lower bound of this set is in the set. The line from P to $\gamma(t_0)$



Let X = the closest-to-P intersection with S of the line from P to $\gamma(t_0)$.

Now the line from P to $\gamma(t_0)$ $\subset$ (interior of S) $\cup S$.

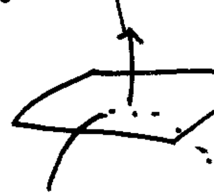
(Otherwise, this line could not be a limit of the lines P to $\gamma(t)$, $t < t_0$, which are all $\subset$ interior of S). It follows that the mean curvature of S at X relative to the exterior normal of S at X cannot be positive: if mean curv. at $X > 0$, then there are no (parts of) lines (even locally) through X that extend both ways from X and lie in $\mathrm{int}(S) \cup S$ (see figure). So mean curvature at $S \leq 0$.

But this contradicts Step 2.

Recall: Gauss map(p) = $\Gamma(p)$ (notation) = exterior unit normal at p.

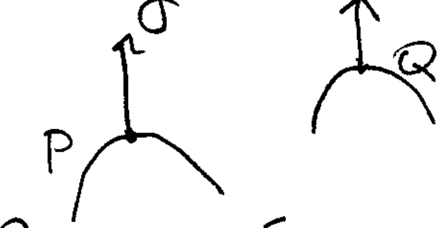
Step 4: The Gauss map of S is one-to-one, onto the unit sphere.

Proof: onto: Given V_0, a unit vector, consider the function $F_{V_0} : S \to \mathbb{R}$ defined by $F_{V_0}(p)$ $\langle p, V_0 \rangle$. Let p be a maximum on S for F_{V_0}. Then exterior unit normal at $p = V_0$. (Detail: Wolog $V_0 = (0,0,1)$ so p = point where z-coordinate is maximum



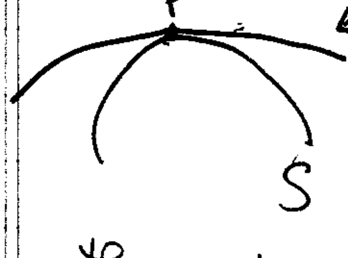
Surface lies entirely under horizontal tangent plane at p so exterior normal $= (0,0,1)$.

One-to-one: Suppose $P, Q \in S$ are such that $N(P) = N(Q)$ i.e. $\Gamma(P) = \Gamma(Q)$. Wolog $\Gamma(P) = \Gamma(Q) = (0,0,1)$.

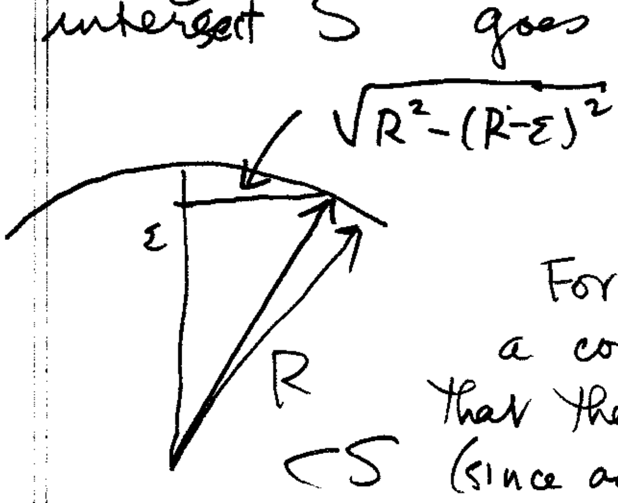


Wolog $z(P) \le z(Q)$ so line from P to Q is horizontal or slopping upward.

Choosen $\varepsilon > 0$ small and let $P_\varepsilon = P - (0,0,\varepsilon)$ $Q_\varepsilon = Q - (0,0,\varepsilon)$. For $\varepsilon > 0$ suffecently small P_ε and Q_ε are both in interior of S. Hence so is the line segment from P_ε to Q_ε by Step 3. However, because $K(P) > 0$, there is a sphere with horizontal tangent plane at P such that near P, $S \cup \text{int} S \subset$ downside (interior) of sphere, as in the figure



(it suffices to take a sphere with radius R so large that $1/R <$ smaller of (abs value of) the two eigenvalues of second fundamental form of S at P). So as $\varepsilon > 0$ goes to 0, the maximum possible length of a horizontal or upward tilting line segment from P that does not intersect S goes to 0. It is length $\le$



$\sqrt{R^2 - (R-\varepsilon)^2}$ as shown in the figure.

For ε small enough, we get a contradiction of the fact that the line from P_ε to Q_ε is $\subset S$ (since as $\varepsilon \to 0$, $\text{dis}(P_\varepsilon, Q_\varepsilon) \to \text{dis}(P, Q) > 0$)

Step 5: $\int_S K\,dA = 4\pi$

(This verifies the Gauss Bonnet Theorem in this case).
Proof: If $p \in S$, the tangent plane at p of S = tangent plane of the unit sphere at $\Gamma(p)$. The differential of the Gauss map Γ is thus a mapping from this tangent plane to to itself. We shall show that

$|$(Jacobian) determinant of this map$| = |J|$

i.e. $|$ Jacobian of $\Gamma|$

is in fact $K(p)$, the Gauss curvature at p. (in general, one has $|J| = |K(p)|$ but $K>0$ in our case: general case Jacobian $J = K$ including signs!)
To check this, wolog $S = (u, v, f(u,v))$ $f_u = f_v = 0$ at $(0,0)$ as usual. Differential of Γ is computed from $\Gamma|_{(u,v)} = \dfrac{(1,0,f_u)\times(0,1,f_v)}{\sqrt{1+f_u^2+f_v^2}}$

$= \dfrac{(-f_u, -f_v, 1)}{\sqrt{1+f_u^2+f_v^2}}$ where denominator makes vector unit length.

Direct computation gives $\left.\dfrac{\partial\Gamma}{\partial u}\right|_{(0,0)} = (-f_{uu}, -f_{uv}, 0)$

$\left.\dfrac{\partial\Gamma}{\partial v}\right|_{(0,0)} = (-f_{uv}, -f_{vv}, 0)$. Jacobian

matrix is $\begin{pmatrix} -f_{uu} & -f_{uv} \\ -f_{uv} & -f_{vv} \end{pmatrix}$. So Jacobian

determinant $= f_{uu}f_{vv} - f_{uv}^2$ = Gauss curvature.
$\int K\,dA = 4\pi$ now follows from change of variables for double integrals together with Γ 1-1, onto, nonsingular