The Intrinsic Gauss Curvature Formula: An Example

In "Local Surface Theory", we derived an intrinsic formula for $L_{11}L_{22} - L_{12}^2$, namely

$$L_{11}L_{22} - L_{12}^2 = -\frac{\partial}{\partial u}\langle S_{uv}, S_v\rangle + \frac{\partial}{\partial v}\langle S_{uu}, S_v\rangle$$

$$+ \langle T(S_{uv}), T(S_{uv})\rangle - \langle T(S_{uu}), T(S_{vv})\rangle$$

We work out the right-hand side explicitly for the case $S(u,v) = (u, v, f(u,v))$ where $f_u = 0 = f_v$ at $(0,0)$ [usual set-up]

$S_u = (1, 0, f_u)$ $\quad S_v = (0, 1, f_v)$

$S_{uu} = (0, 0, f_{uu})$ $\quad S_{uv} = (0, 0, f_{uv})$, $\quad S_{vv} = (0, 0, f_{vv})$

$\langle S_{uv}, S_v\rangle = f_{uv} f_v$ $\qquad \langle S_{uu}, S_v\rangle = f_{uu} f_v$.

Now at $(0,0)$ $\quad f_u = f_v = 0$ and S_{uu}, S_{uv}, S_{uv} are all normal to the surface (so tangent parts $= 0$)

So the right-hand side of the formula for $L_{11}L_{22} - L_{12}^2$ is, at $(0,0)$

$$= -\frac{\partial}{\partial u}(f_{uv} f_v) + \frac{\partial}{\partial v}(f_{uu} f_v) + 0 - 0$$

$$= -f_{uv} f_{vu} + f_{uu} f_{vv}$$

[note that the other Leibniz Rule terms $-f_{uvu} f_v + f_{uuv} f_v$ are 0 since $f_v = 0$ at $(0,0)$]*

$$= L_{11}L_{22} - L_{12}^2$$

(at $(0,0)$) because at $(0,0)$

$N = (0, 0, 1)$ so $\quad L_{11} = -\langle (0,0,1), (0,0,f_{uu})\rangle = -f_{uu}$

$L_{12} = -f_{uv}$ & $L_{22} = -f_{vv}$.

It works!

* They would cancel anyway! (mixed partials $=$)