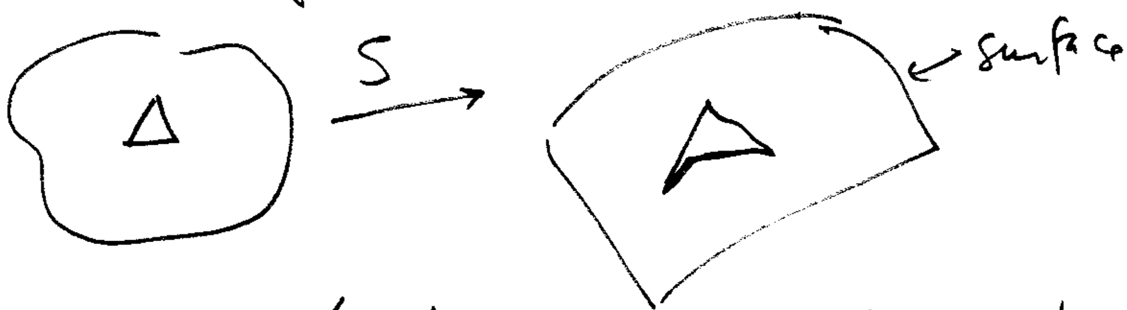
Outline Proof of Abstract Gauss Bonnet Theorem from Gauss Bonnet Formula

We talk ~~first~~ about oriented surfaces only: for nonorientables, do "double cover". (later)

A "triangle" in a surface is the (u,v) parameterization image of a triangle in the (u,v) plane



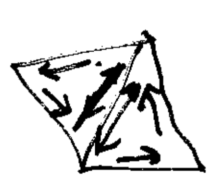
Note that $S(\Delta)$ need not have straight-line edges or be contained in a plane. But it does have "angles" at its "vertices". And it is oriented:

Gauss Bonnet Formula:

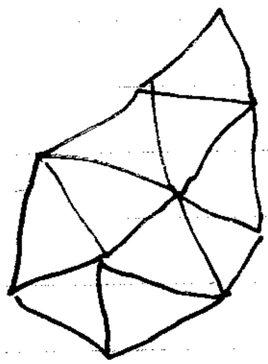
Σ (interior) angles of a triangle in a surface

$$= \pi + \int_{\text{interior of triangle}} K\, dA \qquad + \text{ edge terms}$$

We shall specify what the edge terms are later. But for the moment all we need is that they reverse sign when the orientation of the edges are reserved so that if we add two triangles with a common edge, the ~~two~~ edges ~~are~~ is counted oppositely in the two.



Now suppose we "triangulate" the whole surface



. Then we sum over all triangle. ~~How~~

$$\Sigma \text{ all interior angles} = 2\pi V$$

(V = no. of vertices).

So

$$2\pi V = \pi F + \int K\, dA, \quad F = \text{no. of faces}$$

(edge terms cancel)

So

$$\int K\, dA = 2\pi \left(V - \frac{F}{2}\right)$$

(F = no. of faces)

Now, since each edge belongs to exactly two triangles

$$3F = 2E$$

So

$$F/2 + F = E$$

and

$$+F/2 = +E - F$$

So

$$V - \frac{F}{2} = V - E + F \text{ and}$$

$$\int K\, dA = 2\pi (V - E + F).$$

Since left hand side does not depend on triangulation, neither does right hand side. And since right hand side is "topological", so is left hand side. (Strictly speaking, "topological" means differentiably homeomorphic but this turns out not to matter.