Integration on surfaces and the Gauss-Bonnet Theorem

If $S(u,v)$ is a "surface path", then by definition when f is a function on S (or a function on (u,v) domain)

$$\int_S f = \int_{(u,v) \text{ domain of } S} f(S(u,v)) \cdot \sqrt{EG-F^2}\, du\, dv$$

or, if one thinks of f as a function on the (u,v) domain

$$= \int_{(u,v) \text{ domain}} f(u,v) \sqrt{EG-F^2}\, du\, dv.$$

Now by the Lagrange identity

$$\sqrt{EG-F^2} = \| S_u \times S_v \|$$

so this definition coincides with the one you are familiar with from multivariable calculus.
If $(\hat{u},\hat{v})$ is another parameterization then

$$S_{\hat{u}} = \frac{\partial u}{\partial \hat{u}} S_u + \frac{\partial v}{\partial \hat{u}} S_v$$

$$S_{\hat{v}} = \frac{\partial u}{\partial \hat{v}} S_u + \frac{\partial v}{\partial \hat{v}} S_v$$

so $$\| S_{\hat{u}} \times S_{\hat{v}} \| = \left| \det \begin{pmatrix} \frac{\partial u}{\partial \hat{u}} & \frac{\partial v}{\partial \hat{u}} \\ \frac{\partial u}{\partial \hat{v}} & \frac{\partial v}{\partial \hat{v}} \end{pmatrix} \right| \, \| S_u \times S_v \|$$

by direct calculation. Call $\left| \frac{\partial(u,v)}{\partial(\hat{u},\hat{v})} \right|$.

Then
$$\int f \sqrt{\hat{E}\hat{G}-\hat{F}^2}\, d\hat{u}\, d\hat{v} = \int f \, \| S_{\hat{u}} \times S_{\hat{v}} \| \, d\hat{u}\, d\hat{v}$$

$$= \int f \, \| S_u \times S_v \| \left| \frac{\partial(u,v)}{\partial(\hat{u},\hat{v})} \right| d\hat{u}\, d\hat{v} = \int f \, \| S_u \times S_v \| \, du\, dv$$

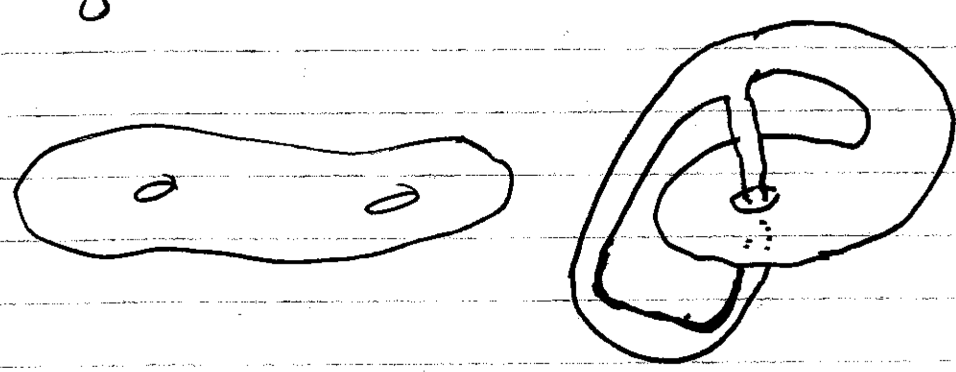
$$= \int f \sqrt{EG-F^2}\, du\, dv$$

so the integral is independent of parameterization of S.

Note that $\sqrt{\hat{E}\hat{G}-\hat{F}^2} = \sqrt{EG-F^2} \cdot \left| \frac{\partial(u,v)}{\partial(\hat{u},\hat{v})} \right|$ could be checked without going through $\|S_u \times S_v\|$ etc., that is using just the transformation properties of E, F and G. So integration is also independent of parameterization on "abstract" surfaces: it is "intrinsic".

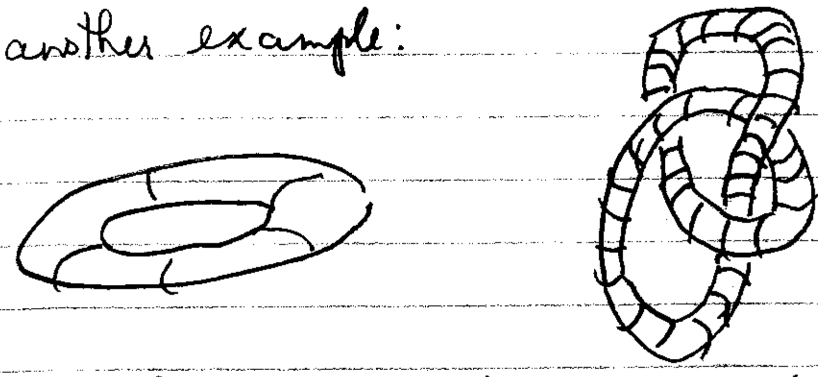
Definition: Two closed surfaces (bounded surfaces with no edges) S_1, S_2 are <u>homeomorphic</u> if there is a continuous function $F: S_1 \to S_2$ which is 1-1 onto and has F^{-1} continuous. (actually, the continuity of F^{-1} is automatic here, but we leave this as an exercise).

Note that S_1 and S_2 can be homeomorphic without S_1 being "deformable" into S_2 within $\mathbb{R}^3$:



Both of these are two-hole surfaces but one of them is "knotted".

Here is another example:



Note: A sphere cannot be "knotted". A (smooth) surface homeomorphic to a sphere can always be deformed into a standard round sphere.

Gauss Bonnet Theorem: If S_1 and S_2 are two closed surfaces (in $\mathbb{R}^3$) then

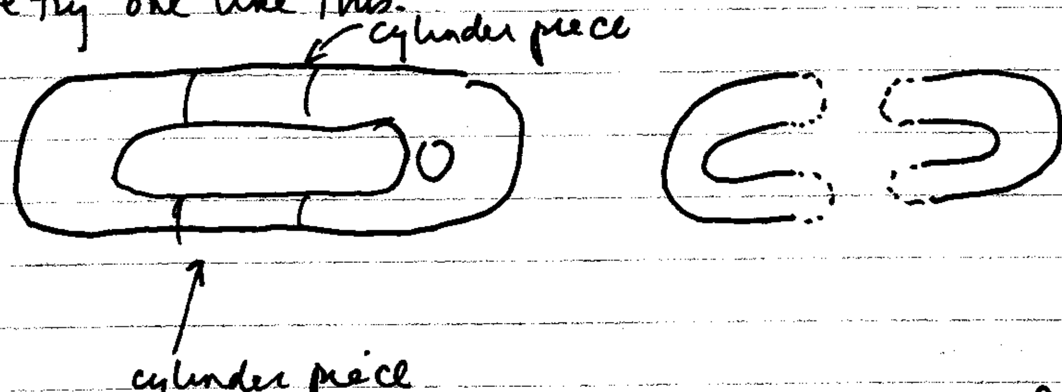
$$\int_{S_1} K_1 \, dA_1 = \int_{S_2} K_2 \, dA_2$$

where K_1, K_2 are the Gauss curvatures and dA_1, dA_2 refer to area integration on S_1, respectively S_2 as already defined.

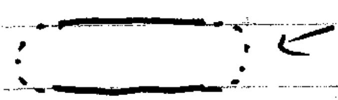
Actually it is not necessarily that the surfaces be in $\mathbb{R}^3$: the theorem is about "abstract" surfaces. But we shall be mostly discussing it for surfaces in $\mathbb{R}^3$.

The Gauss-Bonnet Theorem says that $\int K\,dA$ depends only on the topological type of the surface. It is of course interesting to know what $\int K\,dA$ is for various types.

For a sphere, this is something we already know: $\int K\,dA = \int_{\text{unit sphere}} 1 = 4\pi$. To find the number for a "torus" we can look at any one we like. So we try one like this:



We cut out pieces of cylinder and cap off ends. The caps are not exactly hemispheres (they have to join the cylinders smooth). But $\int K\,dA$ over a cap $= 2\pi$ because $\int K\,dA = 4\pi$ for the sphere shown. So



$$2\int_{\text{cap}} K\,dA = 4\pi$$

So $\int_{\text{cap}} K\,dA = 2\pi$. So $4\int_{\text{cap}} K\,dA + \int_{\substack{\text{torus}\\ -\text{cylinder pieces}}} K\,dA = 2\cdot 4\pi$

since we ended up with two "spheres". But $\int_{\text{cylinder piece}} K\,dA = 0$

So $\int_{\substack{\text{torus}\\ -\text{cylinder pieces}}} K\,dA = \int_{\text{torus}} K\,dA$.

$K \equiv 0$ on cylinder!

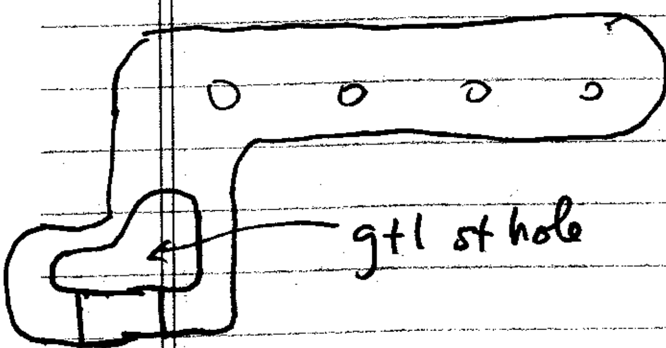
So

$$\int_{\text{torus}} K\,dA = 2\cdot 4\pi - \underset{\text{cap}}{4(2\pi)} = 0$$

Inductively, we see that a "(g+1)-hole" surface S has

$$\int_S K\,dA = (\text{g hole surface integral}) - 4\pi$$

by cutting out cylinder piece and capping.



So g-hole surface has

$$\int K\,dA = 2\pi(2-2g)$$
$$(= 4\pi - 4g\pi)$$

Actually, every closed surface in $\mathbb{R}^3$ is a sphere or a "g-hole" surface for some g.
g is called the <u>genus</u> of the surface.

and what we have been calling informally a g-hole surface is called formally a "compact surface of genus g".

So S = genus g surface has

$$\int_S K\,dA = 2\pi(2-2g) = 2\pi\,\chi(S)$$

where $\chi(S) = 2-2g$ = "Euler characteristic

The proof of the Gauss Bonnet Theorem in complete generality is a quite intricate matter (this will be discussed later, in a separate write-up). But there is one instance where the result can be relatively easily established without its being at all "obvious". Namely, for a closed surface in $\mathbb{R}^3$ with Gauss curvature $K > 0$ everywhere, the result can be established by direct integration methods.

For this, we recall that for such a surface S the "Gauss map" that sends each point $p \in S$ to its (exterior) normal $N(p)$ is a 1-1 onto (C^∞) mapping from S to the unit sphere $S^2 \subset \mathbb{R}^3$. (Homework exercise earlier). The crucial observation for our purposes is that the Jacobian (determinant) of the mapping $\Gamma : S \to S^2$ has absolute value = absolute value of the Gauss curvature (suitably interpreted, Jacobian = Gauss curvature holds without absolute values).

[Recall here that the |Jacobian determinant| = the "area multiplication factor" in the following sense: if p is a point of S, then there is a number A_p associated to p with following property: for each $\varepsilon > 0$ there is an open set U_ε with $p \in U_\varepsilon$ such that for any open set $V \subset U_\varepsilon$, area$(\Gamma(V))$ / area(V) $\in (F_p - \varepsilon, F_p + \varepsilon)$.

(F_ε is f for factor here). This actually comes from the same ideas as the $\|S_u \times S_v\|$ factor in the definition of area integration. If $S(u,v)$ is a local parameterization of S that $(u,v) \to \Gamma'(S(u,v))$, call this $\hat{\Gamma}$ has a number-valued function associated $(u,v) \to \|\hat{\Gamma}_u \times \hat{\Gamma}_v\|$ analogous to $\|S_u \times S_v\|$. In this set-up,

$$F_\varepsilon = \|\hat{\Gamma}_u \times \hat{\Gamma}_v\| / \|S_u \times S_v\|$$

which is easily seen to be independent of the parameterization (u,v).]

It follows from the fact that Γ' is 1-1 onto that

$$\int_S F_\varepsilon \, dA_S = \int_{S^2} 1 \, dA_{S^2} = 4\pi.$$

So the Gauss-Bonnet Theorem for this case will follow if we can show that $F_\varepsilon = |K| = K$, K = Gauss-curvature of S.

This can be shown in generality, but it is in fact sufficient, since everything is "geometric" (invariant under rigid motions), to treat, without loss of generality the situation where S is of the form

$(u,v) \to (u, v, f(u,v))$, $p = (0,0)$,

$S(p) = (0,0,0)$ $\quad f_u = f_v = 0$ at $(u,v) = (0,0)$

In this notation

$$\hat{\Gamma}(u,v) = \frac{[(1,0,f_u) \times (0,1,f_v)]}{\|(1,0,f_u) \times (0,1,f_v)\|}$$

$$= \frac{(-f_u, -f_v, 1)}{(1 + f_u^2 + f_v^2)^{1/2}}$$

The |Jacobian determinant| of $\hat{\Gamma}$ at $(u,v) = (0,0)$ is (using $f_u = f_v = 0$ at $(0,0)$):

$$\left| \det \begin{pmatrix} f_{uu} & f_{vu} \\ f_{uv} & f_{vv} \end{pmatrix} \right| = |f_{uu} f_{vv} - f_{uv}^2| = K \ (>0) \text{ by hypothesis}$$

To check this note that $\hat{\Gamma}_u |_{(0,0)} = (-f_{uu}, -f_{uv}, 0)$

and $\hat{\Gamma}_v |_{(0,0)} = (-f_{uv}, f_{vv}, 0)$

so that $\|\hat{\Gamma}_u \times \hat{\Gamma}_v\| = |f_{uu} f_{vv} - f_{uv}^2|$

$= K$ (since $K > 0$).

This completes the proof.