Example of Pointwise Property for D-formula (all Lie brackets = 0 case)

D formula

$$2\langle D_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle$$
$$+ \langle [X,Y],Z\rangle - \langle [X,Z],Y\rangle$$
$$- \langle X,[Y,Z]\rangle$$

Suppose $[X,Y] = [Y,Z] = [X,Y] \equiv 0$.

We want check "pointwise" ($\Leftarrow$ function linearity) in first the $X$-slot: The terms that arise that do not have $f$ factored out in front are; when $X$ is replaced by $fX$,

(from $Y(f\langle X,Z\rangle)$) $\quad (Yf)\langle X,Z\rangle$

(from $-Z(f\langle X,Y\rangle)$) $\quad -(Zf)\langle X,Y\rangle$

(from $\langle [fX,Y],Z\rangle$) $\quad -(Yf)\langle X,Z\rangle$

$$[fX,Y] = f[X,Y] - Yf\,X$$

(from $-\langle [fX,Z],Y\rangle$ $\quad +(Zf)\langle X,Y\rangle$

$$[fX,Z] = f[X,Z] - (Zf)X$$

Total $= 0$! as required.

But see note at bottom of next page!

To check pointwise in $Z$ (again all $[\ ,\ ] \equiv 0$) look at terms arising when $Z$ is replaced by $fZ$, which do not just have $f$ out front:

(from $X\langle Y, fZ\rangle$ $= X(f\langle Y, Z\rangle)$) $\qquad (Xf)\langle Y, Z\rangle$

(from $Y\langle X, fZ\rangle$ $= Y(f\langle X, Z\rangle)$) $\qquad (Yf)\langle X, Z\rangle$

(from $-\langle \underbrace{[X, fZ]}_{= f[X,Z] + (Xf)Z}, Y\rangle$) $\qquad -(Xf)\langle Y, Z\rangle$

(from $-\langle X, \underbrace{[Y, fZ]}_{f[Y,Z] + (Yf)Z}\rangle$) $\qquad -(Yf)\langle X, Z\rangle$

total $= 0$.

Actually, it does not really help that much here, the assumption that all $[\ ,\ ] = 0$: all it did was get rid of terms that looked like $f[\ ,\ ]$ anyway, terms that were already function linear. Still, maybe it is a little easier.

Note that function linearity in the $X$ and $Z$ variables (3)
(and the Leibniz property in $Y$, which you can
done at end → check similarly) mean that one can really
just do coordinate things since then

$$\left\langle D_{\sum f_j \frac{\partial}{\partial x_j}} \sum g_k \frac{\partial}{\partial x_k}, \sum h_l \frac{\partial}{\partial x_l} \right\rangle$$

$$= \sum_{j,k,l} f_j h_l \left\langle D_{\frac{\partial}{\partial x_j}} \left( g_k \frac{\partial}{\partial x_k} \right), \frac{\partial}{\partial x_l} \right\rangle$$

$$= \sum_{j,l} f_j h_l \left( \sum_k \left( \frac{\partial g_k}{\partial x_j} \left\langle \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \right\rangle + g_k \left\langle D_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \right\rangle \right.\right.$$

So if we know $\left\langle D_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \right\rangle$, we know everything and $\uparrow$ this

$$= \frac{1}{2} \left( \frac{\partial}{\partial x_j} \left\langle \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \right\rangle + \frac{\partial}{\partial x_k} \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l} \right\rangle - \frac{\partial}{\partial x_l} \left\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \right\rangle \right)$$

$$= \frac{1}{2} \left( \frac{\partial}{\partial x_j} g_{kl} + \frac{\partial}{\partial x_k} g_{jl} - \frac{\partial}{\partial x_l} g_{jk} \right).$$

Perhaps this all looks a bit messy. But on the other hand, when some messy calculation works out seemingly as if by magic, then one can assume in general that some important truth is being revealed!

This is surely true in this case.

Here is how the Leibniz Rule $D_X fY = (Xf)Y + fD_XY$ works. Again we assume all $[\ ] = 0$ for (small) convenience & look at terms with $f$ differentiated when we look at formula

for $X, fY, Z$:

| | |
|---|---|
| from $X\langle fY, Z\rangle = X(f\langle Y,Z\rangle)$ | $(Xf)\langle Y,Z\rangle$ |
| from $-Z\langle X, fY\rangle = -Z(f\langle X,Y\rangle)$ | $-(Zf)\langle X,Y\rangle$ |
| from $\langle [X,fY], Z\rangle$, $[X,fY] = f[X,Y] + (Xf)Y$ | $Xf\langle Y,Z\rangle$ |
| from $-\langle X, \underbrace{[fY,Z]}_{f[Y,Z] - (Zf)Y}\rangle$ | $+(Zf)\langle X,Y\rangle$ |
| | total $= 2(Xf)\langle Y,Z\rangle$ ✓ |