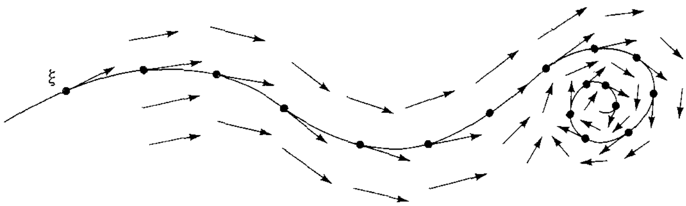
As another application of the Contraction Mapping Theorem, we prove an existence theorem for a first-order system of nonlinear ordinary differential equations. Let U be an open subset of $\mathbb{R}^n$ and let $[a,b]$ be an interval on $\mathbb{R}$. Let F be a continuous function from $U \times [a,b]$ to $\mathbb{R}^n$ and let $\xi \in \mathbb{R}^n$ be a fixed initial value. We consider the initial-value problem of finding a differentiable function $u(t)$ from $[a,b]$ to $\mathbb{R}^n$ that satisfies

$$(8.6) \qquad \begin{cases} \dfrac{du(t)}{dt} = F(u(t),t), & a \leq t \leq b, \\ \\ u(a) = \xi. \end{cases}$$

If the components of u are $u_1,\ldots,u_n$ and those of F are $F_1,\ldots,F_n$, then the initial-value problem becomes

$$\begin{cases} \dfrac{du_j(t)}{dt} = F_j(u_1(t),\ldots,u_n(t),t), & a \leq t \leq b,\ 1 \leq j \leq n, \\ \\ u_j(a) = \xi_j, & 1 \leq j \leq n. \end{cases}$$

Consider, for instance, the special case in which F is independent of t, that is, F is a function from U to $\mathbb{R}^n$. Such an F is called a *vector field* on U. It is visualized



Integral curve of a vector field

by attaching to each $x \in U$ an arrow based at x with direction and length given by $F(x)$. A solution $u(t)$ of (8.6) can be regarded as a curve in U beginning at ξ, with the property that the tangent vector du/dt of the curve at the point $u(t)$ coincides with the value $F(u(t))$ of the vector field F at $u(t)$. The curve $u(t)$ is said to be an *integral curve* of the vector field F.

In order to obtain an existence theorem, we make some hypotheses on F. Choose $r > 0$ so that the closed ball $\{|x - \xi| \le r\}$ centered at ξ with radius r is contained in U. We assume that F is continuous, so that in particular

$$M = \sup\{|F(x,t)| : |x - \xi| \le r,\ a \le t \le b\} \tag{8.7}$$

is finite. We also assume that there is a constant $c > 0$ such that

$$|F(x,t) - F(y,t)| \le c|x - y|, \qquad x,y \in B(\xi;r),\ a \le t \le b. \tag{8.8}$$

Such a condition is called a *Lipschitz condition*; we say that F satisfies a *Lipschitz condition in the first variable*. With these hypotheses, the theorem we obtain is the following.

8.4 Theorem *(Cauchy-Picard Existence Theorem):* Suppose that F and ξ are as above, so that (8.7) and (8.8) hold. Then there exists β, $a < \beta \le b$, such that the initial-value problem (8.6) has a unique solution $u(t)$ defined on the interval $a \le t \le \beta$. Furthermore, β depends only on the parameters r, M, and c.

Proof: The function $u(t)$ satisfies the initial-value problem (8.6) on an interval $[a,\beta]$ if and only if it satisfies the integral equation

$$u(t) = \xi + \int_a^t F(u(s),s)ds, \qquad a \le t \le \beta. \tag{8.9}$$

In turn, u is a solution of (8.9) if and only if it is a fixed point of the integral operator Φ, defined by

$$(\Phi u)(t) = \xi + \int_a^t F(u(s),s)ds, \qquad a \le t \le \beta.$$

We must specify carefully the domain of the operator Φ.

Fix β, $a < \beta \le b$, and let E be the set of continuous functions $u(t)$ from the interval $[a,\beta]$ to $\mathbb{R}^n$ that satisfy

$$|u(t) - \xi| \le r, \qquad a \le t \le \beta.$$

Endowed with the metric of uniform convergence

$$d(u,v) = \sup\{|u(t) - v(t)| : a \leq t \leq \beta\}, \qquad u,v \in E,$$

E becomes a complete metric space. We aim to show that if β is chosen sufficiently near a, then Φ is a contraction mapping of E into E.

First we must arrange that $\Phi(E) \subseteq E$. Let $u \in E$. Using (8.7), we obtain

$$\begin{aligned} |(\Phi u)(t) - \xi| &= \left| \int_a^t F(u(s),s)ds \right| \\ &\leq M(t - a) \\ &\leq M(\beta - a). \end{aligned}$$

To place Φu in E, it suffices then to choose β so that

(8.10) $$\beta - a \leq r/M.$$

Now let $u,v \in E$. Using (8.8), we obtain

$$\begin{aligned} d(\Phi u,\Phi v) &= \sup_{a \leq t \leq \beta} |(\Phi u)(t) - (\Phi v)(t)| \\ &= \sup_{a \leq t \leq \beta} \left| \int_a^t [F(u(s),s) - F(v(s),s)]ds \right| \\ &\leq c \sup_{a \leq t \leq \beta} \int_a^t |u(s) - v(s)|ds \\ &\leq c(\beta - a)d(u,v). \end{aligned}$$

In order that Φ be a contraction, it suffices then to choose β so that

(8.11) $$\beta - a < 1/c.$$

Choosing β to satisfy (8.10) and (8.11) and applying the Contraction Mapping Theorem (Theorem 8.1), we obtain the existence assertion for the Cauchy-Picard Theorem. To prove the uniqueness, observe that any solution of the initial-value problem (8.6) must lie in the space E defined above, provided β is chosen sufficiently near a. By the uniqueness assertion of the Contraction Principle, the solution must coincide with the solution produced above, at least in some small interval $[a,a + \varepsilon]$. Applying this local uniqueness assertion to each point of $[a,\beta]$, we deduce easily that the solution is unique. □

Actually the above proof contains information that goes substantially beyond the statement of the theorem. It provides us with a concrete iteration procedure for approximating the solution of (8.6), and the estimate (8.3) yields a specific bound on the error of the approximation. By keeping track of the error terms, one can prove, for instance, that if F and ξ depend continuously on some other parameters, then the solutions of (8.6) also depend continuously on those parameters. For instance, the solutions of (8.6) depend continuously on the initial condition ξ. We return briefly to the abstract situation, in order to give an indication of how one might approach the problem of dependence of solutions on parameters.