Integral Curves of Vector Fields

If M is a C^∞ manifold and V a C^∞ vector field on it, and if p is a given point of M, then one can ask if there is a C^∞ curve $\gamma : (a,b) \to M$, some $a < 0 < b$ with

$$V(\gamma(t)) \left[= \text{"}\frac{d\gamma}{dt}\text{"} = \gamma'(t) \right] = \text{] tangent}$$

vector of γ for all $t \in (a,b)$.

Such a curve is called an integral curve of V with initial point p.

In local coordinates, for $|t|$ small, coordinates around p, we can write $(x_1(t), \ldots x_n(t)) = \gamma(t)$.

And the integral curve condition becomes

$$V_i(x_1(t), \ldots, x_n(t)) = \frac{dx_i}{dt}$$

where $V = \sum V_i \frac{\partial}{\partial x_i}$ defines the V_i.

A standard theorem about ordinary differential equations (Picard Existence and Uniqueness Theorem cf. T. Gamelin and R. Greene, Introduction to Topology, Chapter I where this is derived from contraction mapping considerations) gives:

$\exists\ \varepsilon > 0$ ~~such that~~ and a neighborhood U of p such that for all $q \in U$ there are C^∞ curves $\gamma_q : (-\varepsilon, \varepsilon) \to M$ with γ_q an integral curve of V. Moreover if $\sigma_q : (a,b) \to M$, $a < 0 < b$,

is an integral curve of V with $\sigma_q(0) = q$ then $\sigma_q(t) = \gamma_q(t)$ for all $t \in (a,b) \cap (-\varepsilon, \varepsilon)$.
Finally, the function

$$(-\varepsilon, \varepsilon) \times U \to M$$

defined by $(t, q) \to \gamma_q(t)$ is C^∞ on $(-\varepsilon, \varepsilon) \times U$, that is simultaneously C^∞ in $t \in \mathbb{R}$ and $q \in M$.

From uniqueness here, it follow easily that for each $p \in M$ there is a maximal open subset of $\mathbb{R}$ of the form (a, b)
$a < 0 < b$ (a can be $-\infty$, b $+\infty$)
such that there is a C^∞ curve $\gamma : (a,b) \to M$ with $\gamma(0) = p$ and γ an integral curve of V.
("The integral curve" hearafter means maximal one)
Exercise: If M is compact, this maximal integral curve is always defined on $(-\infty, +\infty)$.
{ If all integral curves of V are defined on $(-\infty, +\infty)$, then one says V is "complete".

Definition: The flow of a vector field V is the function $\varphi_t(q)$ defined by $\varphi_t(q)$ = the point of M given by $\gamma_q(t)$, where γ_q is the (maximal) integral curve of V with initial point q — whenever $\gamma_q(t)$ is defined!

Examples: $V(x)$

(1) $M=\mathbb{R}^2$, $V=\frac{\partial}{\partial x}$, $\varphi_t(q) = (\vec{q} + t(1,0)) \in \mathbb{R}^2$

(1a) $M=\mathbb{R}$ $V = \partial/\partial x$ $\varphi_t(q) = q+t$

(2) $M=\mathbb{R}$, $V(x) = x\frac{\partial}{\partial x}$ $\gamma_p(t) \equiv 0$ if $p=0$,

$\gamma_p(t) = pe^t$ if $p \neq 0$ (of course, this is consistent with this)

Note that these are complete (even though $\mathbb{R}$ is not compact).

(3) $M=\mathbb{R}$ $V(x) = x^2 \frac{\partial}{\partial x}$. Suppose $x(t)$ is an integral curve. Then integral curve at 0 = constant 0, while if not at 0, $x^2(t) = x'(t)$ so $x'(t)/x^2(t) = 1$. Integrating gives $-\int \frac{d}{dt}(1/x(t))dt = t$. Or

So $C - \frac{1}{x(t)} = t$ or $x(t) = \frac{1}{C-t}$

(This works $\frac{d\,x(t)}{dt} = \frac{-1}{(C-t)^2} \cdot -1 = \left(\frac{1}{C-t}\right)^2$).

For initial point $p \neq 0$, we have $C = \frac{1}{p}$ (for $p=0$, int curve is constant 0 as noted).

Notice that integral curves are not complete when t gets to C, one has a "singularity".

For instance, look at $\gamma_1 = \frac{1}{1-t}$.

As $t \xrightarrow{\text{tries}} +\infty$ we encounter trouble with $t \to 1^-$: If t is <1 but close 1, $\frac{1}{1-t}$ is large positive: "at" $t=1$, we are already "at $+\infty$".

As $t \to \infty$, $\frac{1}{1-t} \to 0$. So the initial point $= 0$ integral sits at 0, the curves with positive initial point "get to $+\infty$ in finite positive time" but go back to 0 with negative time ($t \to -\infty$) without ever reaching 0. If $p < 0$, ~~then~~, then $\varphi_p(t)$ goes to 0 without ever reaching it as $t \to \infty$ but "gets to $-\infty$" by negative time p. (Exercise). This makes sense geometrically in that V always points to the right. So things come in from $-\infty$ very fast but slow up without ever quite making it to 0, 0 just sits there, things on the right go fast off to $+\infty$ but took forever to go from (arbitrarily close to) 0 to get to their initial point.

Observation: $\varphi_{t_1}(\varphi_{t_2}(p)) = \varphi_{t_1+t_2}(p)$ (when defined)
Proof: Both sides are with t_2 fixed integral curves in t_1 as variable with initial point $\varphi_{t_2}(p)$.

Exercise: Check this works for the $x^2 \frac{\partial}{\partial x}$ flow where curves are $1/(C-t)$, $C =$ initial point.

(4) $M = \mathbb{R}^2 \quad V(x,y) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \text{"}\frac{\partial}{\partial \theta}\text{"}$

$\gamma_0(t) \equiv 0$

$\gamma_p(t) = ((\cos t)x_0 - (\sin t)y_0, (+\sin t)x_0 + (\cos t)y_0)$

if $p = (x_0, y_0)$

(Check this).

V is the "infinitesimal generator" of rotation around the origin counterclockwise.

Note that if V is complete, the flow φ_t, defined everywhere for all t, is a group (isomorphic to additive group of $\mathbb{R}$): $\varphi_1 \circ \varphi_2 = \varphi_{t_1+t_2}$, abelian group.

(5) $M = \mathbb{R}^2 - \{\vec{0}\}$, $V(x,y) = \frac{x}{\sqrt{x^2+y^2}}\frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial}{\partial y}$

This is "$\frac{\partial}{\partial r}$" (unit radial vector field").

Flow is $(r,\theta) \to (r+t, \theta)$ in polar coordinates

or

$\varphi_t(x,y) = (x,y) + \left(\frac{tx}{\sqrt{x^2+y^2}}, \frac{ty}{\sqrt{x^2+y^2}}\right)$

Check $\frac{d}{dt}\varphi_t(x,y) = \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right)$.

Note that the flows "commute": One in polar coord is $(r,\theta) \to (r, \theta+t)$, other in polar coord is $(r,\theta) \to (r+t, \theta)$.

So $\varphi_t(\psi_s((r,\theta))) = \psi_s(\varphi_t((r,\theta)))$

if φ_t is flow (4), ψ is flow (5).

On the other hand, we already know that $[\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}] \equiv 0$. This suggests that

maybe there is some relationship between $[V,W] \equiv 0$ and flow of V commuting with flow of W. (Pretty slim evidence — but anyway it's true). This is correct:

Theorem: If V, W are complete vector fields on M with flows φ_t and ψ_t then for all $p \in M$

$$\varphi_t(\psi_s(p)) = \psi_s(\varphi_t(p))$$

Actually, completeness is not needed. The result is local in a sense. All you need is flows defined on a rectangle in t, s. This will be apparent in the proof.

Preliminary Lemma: If $V(p) \neq \vec{0} \in T_pM$, then $\exists$ a local coordinate system $(x_1, \ldots x_n)$ around p with $V \equiv \frac{\partial}{\partial x_1}$ on the (open) domain of the coordinate system.

Why we want this will become apparent later.

Proof: Choose loc. coord. $(y_1, \ldots, y_n)$ such that $V = \frac{\partial}{\partial y_1}$ at p. Then define

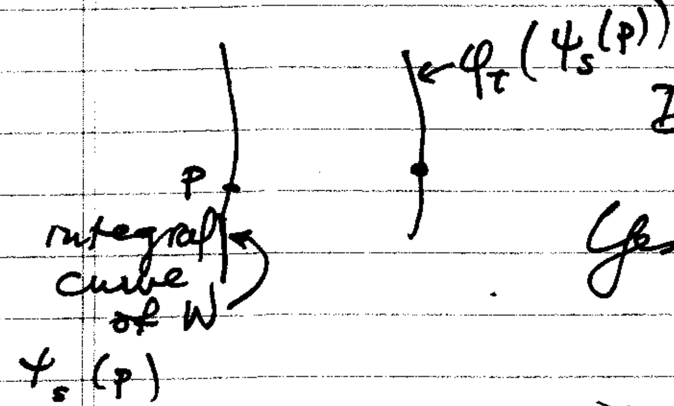
$(x_1, \ldots, x_n) \longrightarrow \varphi_{x_1}($ point with $y_1 = 0$, $y_2 \ldots y_n =$ resp. $x_2, \ldots, x_n)$

By Inverse Func. Theorem this gives local coordinates. $V \equiv \frac{\partial}{\partial x}$ is clear. $\square$

Returning now to the situation of the theorem, we look at the following item: Given $p \in M$, we can define at vector $W_{t,p} \in T_pM$ by

$$W_{t,p} = (\varphi_{-t})_* \left(W(\varphi_t(p)) \right).$$

If $W_{t,p}$ is constant independent of t for all $p \in M$, then the φ_t image of an integral curve ψ_s of W is again an integral curve of W. This is just a matter of chasing where things are:



Does $\frac{d}{ds}\left(\varphi_t(\psi_s(p))\right) \overset{?}{=} W(\varphi_t(p))$

Yes if $(\varphi_t)_* W(p)$

$= W(\varphi_t(p))$

or equiv if $W(p) = (\varphi_t)_*^{-1} W(\varphi_t(p))$

$\checkmark \quad = \underset{(W\text{ int. curve})}{(\varphi_{-t})_*} \underset{(W\text{ int curve})}{W(\varphi_t(p))}$

But if φ image of ψ-curve is a ψ-curve then flows commute.

So now we just need $W_{t,p}$ constant.

For this, we need only to compute $\frac{d}{dt} W_{t,p} \equiv 0$ all t

(when $[V,W]=0$). Now

$$\frac{d}{dt} W_{t,p}\Big|_{t_0} = \lim_{t\to 0} \frac{1}{t}\left((\varphi_{-t_0-t})_* W(\varphi_{t+t_0}(p)) - (\varphi_{-t_0})_* W(p)\right)$$

$$= \lim_{t\to 0} (\varphi_{-t_0})_* \left[\frac{1}{t}\left((\varphi_{-t})_* W(\varphi_{t_0}(p)) - W(\varphi_{t_0}(p))\right)\right]$$

$$= (\varphi_{-t_0})_* \lim \frac{1}{t}\left[((\varphi_{-t})_* W(\varphi_{t_0}(p)) - W(\varphi_{t_0}(p))\right]$$

So in fact we need only check $\frac{d}{dt} W_{t,p}\Big|_{t=0} \equiv 0$ all p ($t=0$ only).

(Here we have used φ_{-t-t_0} = composition of φ_t and φ_{-t_0}).

Let us consider case first that $V(p)=0$. Then in some local coordinates around p $V \equiv \frac{\partial}{\partial x_1}$. If $[V,W]\equiv 0$ then

when $V=\frac{\partial}{\partial x_1}$ we get that W has coefficients that are independent of x_1 (just compute from earlier coordinate formula for $[\,,\,]$).

But if W has coefficients independent of x_1, then the V-flow (which is x_1 translation) leaves W invariant and $\frac{d}{dt} W_{t,p} \big|_{t=0} = 0$. ✓

Now consider a general V. If $V \equiv 0$ in a neighborhood of p, then

$$\frac{d W_{t,p}}{dt} \Big|_{t=0} = 0 \text{ is obvious.}$$

If V is a limit of points q_j with $V \neq 0$,

then $$\frac{dW_{t,p}}{dt}\Big|_{t=0} = \lim_{j \to \infty} \frac{d W_{t,q_j}}{dt}\Big|_{t=0}$$

since $\frac{d W_{t,q}}{dt}\Big|_{t=0}$ is a smooth function of q.

But $$\frac{d W_{t,q_i}}{dt}\Big|_{t=0} \equiv 0 \quad \text{so} \quad \frac{d W_{t,p}}{dt}\Big|_{t=0} = 0.$$

This completes the proof. □

Corollary of the Theorem: If V_i, $i = 1, \ldots n$ are linearly independent at p and $[V_i, V_j] \equiv 0$ near p, all $i,j = 1, \ldots n$, then $\exists$ $(x_1, \ldots x_n)$ loc coords around p $\ni$ $V_i \equiv \frac{\partial}{\partial x_i}$ in a nbhd of p, each $i = 1, \ldots n$.

Proof: Set $(x_1, \ldots x_n) \to$
$$\varphi^1_{x_1}(\varphi^2_{x_2} \cdot \cdot (\varphi^n_{x_n}(p))$$
where φ^i = flow of V_i.
Then $\frac{\partial}{\partial x_1} = \frac{\partial}{\partial x_1}(\varphi^1_{x_1}(\cdot \cdot \cdot (p)) = V_1$

Part commutative property means we can rearrange order & put any $\varphi^i_{x_i}$ in front to get $\frac{\partial}{\partial x_i} = V_i$. $\square$