Hermitian Yang Mills Metrics on Vector Bundles and Stability

Previously, we discussed the application, to Innoue surfaces in particular, of the idea that a stable holomorphic vector bundle admits a Hermitian Yang Mills metric, or what is often called a Hermitian Yang Mills metric. This means by definition an Hermitian metric with the property that the curvature of the associated type (1,0) connection, considered as being a 2-form with values in the endomorphisms of the fibres, has the property that the manifold trace of these endomorphisms is a multiple of the identity. (Precise and explicit formulae for this definition in words will follow shortly).

At first sight, it seems unclear how stability as we defined it earlier is related to this Hermitian Yang Mills condition. The connection in one direction is given by the result of Kobayashi(the other, deeper direction has been discussed already):
If a holomorphic vector bundle E over a compact Kähler manifold M admits a Hermitian Yang Mills metric, then E is semi-stable in the sense that for any holomorphic sub-bundle E' of E, one has that

deg(E') /rank(E') $\leq$ deg(E) /rank(E)

where deg(E')=$\int c_1(E') \wedge \omega^{n-1}$ and similarly for E. Here ω is the Kähler form.

(This actually holds for any subsheaf with the Chern class idea extended suitably).
Moreover, equality holds if and only if E' is direct summand of E in the holomorphic sense, namely that there is another holomorphic sub-bundle E'' such that
E is holomorphically isomorphic to the direct sum of E' and E''.

This result appears startling at first sight. But it turns out to be essentially computational and ,moreover, to be related to a familiar computational fact about complex submanifolds. Namely, it is well-known that the usual Gauss-Codazzi equations for submanifolds have a special form in the holomorphic submanifold case and that this special form implies that a holomorphic sectional curvature of the submanifold is less than or equal to the corresponding holomorphic sectional curvature of the containing manifold, with equality implying the vanishing of (part of) the second fundamental form. The bundle statement just given is in effect thus a bundle version of the submanifold situation.

To make this more detailed, we set up some notation, essentially following Kobayashi's book Differential Geometry of Complex Vector Bundles.(Note what we are calling Hermitian Yang Mills metrics, are called Hermitian Einstein metrics in this reference). Let E' be a holomorphic sub-bundle of a holomorphic vector bundle E over a manifold with Hermitian metric. We write h for an Hermtian metric on E ,which of course induces a metric also denoted by h on E'. We suppose that the underlying complex manifold M is given an Hermitian metric g. Now if s is a local holomorphic section of E', and hence of E, then we can consider the covariant derivatives (along some vector) of s with respect to the Hermitian (1,0) connection of E' and with respect to the Hermitian (1,0) connection of E. This could well be different! It is straightforward to see that the difference between these two items is an element of the fibre of E (at a given point of M with s defined in a neighborhood of the point) which is perpendicular to the fibre of E' at that point in the metric h. This difference is by definition the "second fundamental form" of E' in E with respect to h.

More explicitly , if we write $As= D^{E}s - D^{E'}s$ then A is "pointwise" in s, that is at a point it depends only on the value of s at the point, and As , considered as operating on vectors, is a type (1,0) form. In other words, the second fundamental form A is a (1,0) form with values in the homomorphisms (at each point) from the fibre of E' to the orthogonal complement of the fibre of E' in the fibre of E. This is clearly analogous to the usual second fundamental form idea of submanifold theory . (Note that the orthogonal complement of E' in E , fibre by fibre, yields a complex vector bundle but it is not in general a holomorphic bundle).

Let us write this out in local frames. Let $e_1 , \ldots. e_p$ be a unitary local frame for E' and $e_{p+1} ,\ldots e_r$ be a completion of that frame to a unitary frame for E. Let $\theta_1 ,\ldots.\theta_n$ be a unitary type (1,0) coframe on M. For the index a lying in the set 1,…p, we write

$$A\, e_a = \Sigma\, \omega_a^{\lambda}\, e_\lambda$$

thus defining the form ω_a^{λ}. We also write $\omega_a^{\lambda} = \Sigma\, A_{ai}^{\lambda}\theta_i$ where i is a manifold index, running from 1 to n.
This equation thus defines the A's. Then writing R for the curvature of the sub-bundle E' and S for the curvature of E one obtains(after a moderately extended but straightforward calculation), again using i and j to denote manifold indices, a and b to denote frame indices and λ a summed frame index

$$R_{ai\bar{j}}^{b} = S_{ai\bar{j}}^{b} - \sum A_{ai}^{\lambda}\overline{A_{bj}^{\lambda}}$$

where b ,along with a, lies in the set 1,…,r.

We turn now to the comparison of the degrees of the vector bundles E' and E, where degree was defined above(in the second paragraph). Note first that the form $c_1(E)$ (and similarly for E') is obtained as follows: first one forms the trace with respect to the Hermitian metric on the bundle of the curvature forms with values in the endomorphisms of E to E, so that one obtains

$\Omega_{i\bar{j}} = \sum S^{\alpha}_{\alpha i\bar{j}}$ where the sum is over all e indices α from 1 to r. (For E', the sum would be from 1 to p, with S replaced by R).
Then the form $c_1(E')$ is given by (up to multiplicative constants not dependent on the bundle, which we shall ignore from here on) $\sum \Omega_{i\bar{i}}$. (Note that since we are using unitary frames and coframes, no raising and lowering of indices is involved in these processes). A similar calculation applies to E' as noted.

Now suppose that the metric on the bundle E is Yang Mills Hermitian, so that forming the bundle trace is forming the trace over a diagonal matrix with constant say c on the diagonal. Namely in this case the endomorphism valued curvature form curvature is writable as cαId, where α is a 2-form on the manifold. In this case, one obtains that the degree of E as defined above is r c $\int \alpha \wedge \omega^{n-1}$ (up to the usual fixed constant factors).

In this case, using the formula relating the curvature of E' and that of E and the fact that the double trace of the (subtracted) ΣAA term is nonnegative yields that

deg(E') $\leq$ pc $\int \alpha \wedge \omega^{n-1}$

From this one obtains that

deg(E') /rank(E') $\leq$ deg(E)/rank(E)

with equality if and only if A is identically 0. But in this case, the orthogonal complement of E' in E is a holomorphic sub-bundle(in effect, the fibres of the sub-bundle E' form a parallel sub-bundle of E with respect to its type(1,0) Hermitian connection, which makes their orthogonal complements also parallel and hence a holomorphic sub-bundle).

This picture explains in the sub-bundle case the relationship between being Yang Mills Hermitian(or Hermitian Yang Mills as we have called it earlier) and (semi) stability.[Stability involves by definition strict inequality in the deg/rank inequality displayed above, when E' is a proper sub-bundle]. The extension to the subsheaf situation is obtained by usual sheaf theoretic methods involving resolutions and the usual rules for Chern classes and so on. (cf Kobayashi's book for details). But the sub-bundle picture captures the essential point.

Yang Mills –Hermitian vector bundles, that is holomorphic vector bundles that admit an Yang Mills Hermitian(Hermitian Yang Mills) metric have special properties not obviously related to stability. In particular, one can compute using the standard representations of Chern classes in relationship to curvature that if E is an Hermtian vector bundle of rank r r over a compact Hermitian manifold of dimension n with Kähler form ω and if E with metric h is Hermitian Yang Mills then

$$\int \{(r-1)c_1(E,h) - 2rc_2(E,h)\} \wedge \omega^{n-2} \leq 0$$

and equality implies that (E,h) is projectively flat(that is , the contracted curvature tensor accounts for all the curvature so that $R^{\alpha}_{\beta i \bar{j}} = (1/r)\delta^{\alpha}_{\beta} R_{i\bar{j}}$ for the $R_{i\bar{j}}$ obtained by taking the fibre trace, in our previous notation).

Here c_i (E,h) denotes the ith Chern form for E obtained from the metric h. (In case ω is balanced in the sense already discussed this depends only on the cohomology classes of the forms, that is , on the Chern classes themselves).
(This inequality was further suggestive of the relationship of Hermitian Yang Mills and (semi) stability since this inequality had been shown by Bogomolov to hold for semi-stable vector bundles over algebraic surfaces before the work of Uhlenbeck-Yau and Li-Yau on Hermitian Yang Mills metric on stable bundles.)