# Inverse and Implicit Function Theorem Problems

1. Use Contraction Mapping Ideas for
$f(x) = e^x - 1$, $f(0)=0$, $f'(0)=1$
to find the 1st few terms of the power series of the inverse function. Compare to the Taylor serie of $\ln(y+1)$.

2. Suppose $F: \vec{0} \in$ open set in $\mathbb{R}^2 \to \mathbb{R}^3$, $F(u,v)$ $(u,v) \in \mathbb{R}^2$
and $DF|_{\vec{0}}(1,0)$ and $DF|_{(0,0)}(0,1)$
are independent vectors in $\mathbb{R}^3$. Suppose
$DF|_{\vec{0}}(1,0) \times DF|_{\vec{0}}(0,1)$ has a nonzero z-component.
Prove $\exists\, f: \mathbb{R}^2 \to \mathbb{R}^3$ such that the image of F near $\vec{0}$ $= \{(x,y,f(x,y)) : x,y \text{ near } 0\}$
i.e. the surface "image of F" is locally a graph.

3. Prove: $F: \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 2\} \to \mathbb{R}^2$
is differentiable with continuous DF
and that $F(\vec{0}) = \vec{0}$, DF is nonsingular everywhere and $\|F(x,y)\| \geq 1$ for all $(x,y) \ni \|(x,y)\|=1$. Show $F(\{(x,y): x^2+y^2<1\}) \supset \{(x,y): \|(x,y)\|<1\}$. (Suggestion:
$F(\{\ \})$ is open. Show it is "closed" in $\{(x,y): \|(x,y)\|<1\}$).

4. Show (in detail) how the Inverse Function Theorem $\Rightarrow$ Implicit Function Theorem

5. A function $(x,y) \xrightarrow{F} (u(x,y), v(x,y))$ is holomorphic (by definition) if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Show (assuming any differentiability you like) that $F$ is Locally 1-1 onto at $p$ $\Longleftrightarrow$ $\left\| \left(\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}\right)\Big|_p \right\| \neq 0$.

6. Find the equations in $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$ that correspond to $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ (together). Make this look as nice as possible!

not an InvFuncTh problem!

7. Use the polar coord. formula for $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ $(= \Delta)$ to show that $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ with $\Delta f = 0$ and with "$f$ depending only on $r$" has the form $f(r) = A \log r + B$. Deduce that if $f$ is bounded on $\mathbb{R}^2$ then $f$ is constant.

# Metric Space Topology Exercises

1.(a) Define what it means for a metric space to be connected

(b) Prove that $X \times Y$ is connected if $X$ and $Y$ are connected (see problem 2 for definition of $X \times Y$

2. Given metric space $(X, d_X)$ and $(Y, d_Y)$, define $D : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by

$$D((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

Show that $D$ is a metric on $X \times Y$.

Do the same for $D = \max(d_X(x_1, x_2), d_Y(y_1, y_2))$ and $D = (d_X^2(x_1, x_2) + d_Y^2(y_1, y_2))^{\frac{1}{2}}$.

3.(a) Show that $\sum_{j=1}^{+\infty} \frac{1}{2^j} |x_j - y_j|$ is a metric on the set of all sequences with values in $[0, 1]$

e.g. $(x_1, x_2, x_3 \ldots), (y_1, y_2, y_3 \cdots)$

(b) Prove that convergence in this metric is equivalent to convergence in each "slot", i.e. $\{X_i^{(n)}\} \to \{X_i^{0}\}$ if and only if for each fixed $i$, $x_i^n \to x_0^{i}$ as $n \to +\infty$.

4. Prove that if $(X, d)$ is sequentially compact and $f : X \to Y$ is continuous and onto then $Y$ is sequentially compact.

5. Suppose $X$ is a sequentially compact metric space. Prove: Given $\varepsilon > 0$, there is a finite set $p_1, \dots, p_k \ni d(p_i, p_j) \geq \varepsilon \quad \forall i, j \quad i \neq j$ and $\forall x \in X$, $\exists \; p_j \ni d(x, p_j) < \varepsilon$.
[Such a set is called an $\varepsilon$-net]
(Suggestion: Pick $p_1$ arbitrarily. Pick $p_2 \ni d(p_1, p_2) \geq \varepsilon$. Keep doing this. Show process terminates]

6. Let $S_n$ be an $\varepsilon = 1/n$ net in $X$ ($X$ as in prob 5). Show $\cup S_n$ is a dense set in $X$ (every point in $x$ is a limit of a sequence in $\cup S_n$). Note that $S_n$ is countable.

7. Use problem 6 to show that every covering of $X$ has a countable subcover (imitate euclidean space argument with "rational balls" replaced by $\{ B(s, r) : r \text{ rational}, s \in \cup S_n \}$).

8. Using prob. 7, conclude that covering compactness of a metric space $X$ is implied by sequential compactness.

9. Imitate Euclidean proof to show covering compactness $\Rightarrow$ Seq. compactness for metric spaces

10(a) Show $f: X \to Y$, $X, Y$ metric space is continuous $\iff$ $f^{-1}(V) \subset X$ is open in $X$ for all open sets $V \subset Y$. (b) Do same with "open" replaced by "closed".

11. Show that if $X$ is (covering) compact and $f: X \to Y$ is onto then $Y$ is (covering) compact.

12. Do prob 11 with "compact" replaced by "connected".

13. Prove: If $f: X \to Y$ is continuous and 1-1 onto and if $X$ is compact, then $f^{-1}$ is continuous. [Suggestion: $(f^{-1})^{-1}$(closed) is closed $\iff$ $f$(closed) is closed. Use 10(b).]

14. Let $X$ be a metric space and $C \subset X$ be a compact subset. Suppose $p \notin C$. $\exists\, q \ni q \in C$ and $d(p,q) = \inf_{x \in C} d(p,x)$. In particular, the inf $> 0$.

15. Suppose $X$ is compact and $C_1, C_2$ are compact sets, $C_1 \cap C_2 = \phi$. Show $\inf_{x \in C_1, y \in C_2} d(x,y) > 0$ and inf is attained.