Problem Set II

1. An open interval $(a, b) \subset \mathbb{R}$ is called <u>rational</u> if a & b are both rational numbers. Prove that every open set $U \subset \mathbb{R}$ is the union of all the rational open intervals contained in it.

2. Suppose A is a set $\subset \mathbb{R}$ and $\{U_\lambda : \lambda \in \Lambda\}$ is open $\ni \bigcup_{\lambda \in \Lambda} U_\lambda \supset A$ (i.e. $\{U_\lambda\}$ is an open cover of A).

Prove that $\exists$ $\lambda_1, \lambda_2, \lambda_3 \ldots \ni \bigcup_{j=1}^{\infty} U_{\lambda_j} \supset A$.

(or $j = 1, \ldots n$, in trivial case that Λ is finite)

"Every open cover has a countable subcover".

Suggestion: Consider the set of all rational open intervals $\ni$ the interval $\subset$ some U_λ. Count these: $V_1, V_2, V_3, \ldots$ (set of all rational intervals is countable so this is possible). Set U_{λ_j} = a U_λ that contains V_j.

3. Let A be an uncountable set $\subset \mathbb{R}$. Show that $\exists$ $a \in A$ which is a "condensation point" of A (means: for each $\varepsilon > 0$, the intersection $(a - \varepsilon, a + \varepsilon) \cap A$ is uncountable).

Suggestion: If not, then choose for each $a \in A$ an $\varepsilon_a > 0 \ni (a - \varepsilon_a, a + \varepsilon_a) \cap A$ is countable. The set $\{(a - \varepsilon_a, a + \varepsilon_a) : a \in A\}$ is an open cover of A. Look at a countable subcover (from problem 2).

4. Prove: If a is a condensation point of an (uncountable) set A, then for each $\varepsilon > 0$, there is a condensation point of A besides a itself in $(a-\varepsilon, a+\varepsilon) \cap A$.

Suggestion: One of the sets $\{x \in A : \frac{\varepsilon}{n} \leq |x-a| \leq \varepsilon/(n-1)\}$ $n = 3, 4, 5 \ldots$ must be uncountable since $(a - \varepsilon/2, a+\varepsilon/2) \cap A$ is uncountable. Apply problem 3 to one of these that is uncountable.

5. Show that for a set $A \subset \mathbb{R}$ the following two properties are equivalent:

(a) $\mathbb{R} - A$ is open (that is, for each $x \in \mathbb{R} - A$, $\exists\, \varepsilon_x \ni (x-\varepsilon_x, x+\varepsilon_x) \subset \mathbb{R} - A$)

(b) If $\{a_j : a_j \in A\}$ is a sequence in A which converges (in $\mathbb{R}$), then $\lim a_j \in A$.

6. A set $A \subset \mathbb{R}$ is said to be dense-in-itself if $\forall\, a \in A$, $\exists$ a sequence $\{a_j : a_j \in A\}$ with no a_j equal to a and with $\lim a_j = a$.

Modify our proof (using nested intervals) that $\mathbb{R}$ is uncountable to prove that every set $A \subset \mathbb{R}$, $A \neq \emptyset$, that is closed and dense-in-itself is uncountable.

7. Show that if A is an uncountable subset of $\mathbb{R}$ then the set of condensation points of A is closed and dense-in-itself (we have already shown it is nonempty).

8. Show that if A is a subset of $\mathbb{R}$ then it is the union of a countable (or finite or empty) set and a closed, dense-in-itself set.
(Suggestion: Prove $A \setminus \{\text{condensation points of } A\}$ is empty, finite, or countable).

Terminology:
A closed, dense-in-itself set is often called a "perfect" set.

9. (Baire Category Theorem) A set $A \subset \mathbb{R}$ is said to be dense (in $\mathbb{R}$) if every point x of $\mathbb{R}$ is the limit of some convergent sequence in A. (Example: $\mathbb{Q}$ is dense in $\mathbb{R}$). Use nested intervals to prove that: If U_j, $j=1,2,3\ldots$ are open sets in $\mathbb{R}$ and each U_j is dense in $\mathbb{R}$, then $\bigcap_{j=1}^{+\infty} U_j$ is dense in $\mathbb{R}$. (Note: $\cap U_j$ may fail to be open, e.g., $U_j = \mathbb{R} - \{r_j\}$ where $r_1, r_2, r_3 \ldots$ is an enumeration of the rational numbers).

10. Use problem 9 to show that the set of irrationals is not a countable union of closed sets.
(Suggestion: This is equivalent to $\mathbb{Q}$ not being a countable intersection of open sets. But if $\mathbb{Q} = \cap V_j$, each V_j is dense and $(\cap V_j) \cap (U_j) = \phi$, U_j's as in Example in problem 9. X of (main) conclusion of problem 9.)

11. Prove: If $f : [0,1) \to \mathbb{R}$ is a continuous function, then $\exists$ a continuous function $F : [0,1] \to \mathbb{R}$ with $F(x) = f(x)$ for all $x \in [0,1)$ if and only if f is uniformly continuous.

(Slogan: A continuous function extends to the closure if and only if it is uniformly continuous).

(Suggestion: For the nonobvious direction, prove that if f is uniformly continuous, then $\{f(x_j)\}$ is a Cauchy sequence if $\{x_j\}$ is a sequence in $[0,1)$ with limit $= 1$).