Arzela-Ascoli Theorem.

Lemma: Let $f_j : [0,1] \to \mathbb{R}$ $j = 1,2,3 \ldots$ be a sequence of functions that are uniformly bounded, i.e., $\exists M \ni |f_j(x)| \leq M$ for $\forall j = 1,2,3 \ldots$ and $\forall x \in [0,1]$. Then $\exists$ a subsequence $\{f_{j_k}\}$ $k = 1,2,3 \ldots$ such that for each rational number $r \in [0,1]$, the sequence $\{f_{j_k}(r)\}$ is a Cauchy sequence.

Note: The uniformity of the bound is not needed. The proof actually applies with just a bound at each point. But we do not need this generality.

Proof: Enumerate the rationals in $[0,1]$ by $r_1, r_2, r_3 \ldots$. The sequence $\{f_j(r_1)\}$ has a convergent subsequence (since it is bounded). Take the corresponding subsequence of $\{f_j\}$. This subsequence has itself a subsequence which coverges at r_2, also (same reason). And that subsequence has a subsequence which coverges at r_3, and so on. Choose a subsequence of $\{f_j\}$ by taking the first element of the first subsequence of f's, the second element of the second subsequence, the third of the third, etc. This subsequence converges at <u>all</u> of the points $r_1, r_2, r_3, \ldots$ $\square$

Definition: A sequence $f_j : [0,1] \to \mathbb{R}$, $j = 1,2,3 \ldots$ (of continuous functions) is <u>uniformly</u> <u>equicontinuous</u> if, for each $\varepsilon > 0$, $\exists \delta_\varepsilon > 0 \ni$ $|x - y| < \delta_\varepsilon$ $\Rightarrow$

$$|f_j(x) - f_j(y)| < \varepsilon \quad \text{for all } j = 1,2,3 \ldots$$

Exercise: Formulate a concept of equicontinuity at x and prove that a sequence equicontinuous at each $x \in [0,1]$ is uniformly equicontinuous on $[0,1]$.

AA2

Arzela-Ascoli Theorem: If $f_j : [0,1] \to \mathbb{R}$, $j = 1,2,3\ldots$ is a uniformly bounded, uniformly equicontinuous sequence of (continuous) functions, then $\exists$ a subsequence $\{f_{j_k}\}$ which is uniformly Cauchy.

[A sequence G_k is uniformly Cauchy if $\forall\, \varepsilon > 0, \exists\, N_\varepsilon \ni m, n \geq N_\varepsilon \Rightarrow |G_m(x) - G_n(x)| < \varepsilon$ for $\forall\, x \in$ domain]

Proof: Choose a subsequence as in the Lemma, say f_{j_k}, $k = 1,2,3\ldots$ that "converges at each rational point of $[0,1]$," i.e., if r is rational, $r \in [0,1]$ then $f_{j_k}(r)$ is a Cauchy sequence. Claim: This subsequence is uniformly Cauchy. To prove this, choose, given $\varepsilon > 0$, a $\delta_{\varepsilon/3} \ni x, y \in [0,1]$, $|x-y| < \delta_{\varepsilon/3}$ $\Rightarrow |f_j(x) - f_j(y)| < \varepsilon/3$, $\forall\, j = 1,2,3\ldots$ (uniform equic. here!)

Choose L a positive integer $\ni 1/L < \delta_{\varepsilon/3}$ and consider the finite set of rational numbers $0, 1/L, 2/L, \ldots L/L = 1$.

Since f_{j_k} converges (is Cauchy) at each rational number $\exists\, N_\varepsilon \ni m, n \geq N_\varepsilon \Rightarrow \left| f_{j_m}\!\left(\frac{p}{L}\right) - f_{j_n}\!\left(\frac{p}{L}\right) \right| < \varepsilon/3$ for $\forall p = 0, 2 \ldots L$.

(just take $N =$ max of $N_0, \ldots, N_L$ where N_p "works" at $\frac{p}{L}$).

Now for each $x \in [0,1]$, $\exists\; p \ni \left|x - \frac{p}{L}\right| < \delta_{\varepsilon/3}$. For this p, and $m, n \geq N_\varepsilon$

$$|f_{j_m}(x) - f_{j_n}(x)| \leq \left|f_{j_m}(x) - f_{j_m}\!\left(\tfrac{p}{L}\right)\right| + \left|f_{j_m}\!\left(\tfrac{p}{L}\right) - f_{j_n}\!\left(\tfrac{p}{L}\right)\right| + \left|f_{j_n}\!\left(\tfrac{p}{L}\right) - f_{j_n}(x)\right|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon. \quad \square$$