On the Uniform Continuity of Crossing Probabilities

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I will talk about some uniform continuity of crossing probabilities for 2D critical percolation. This is a (perhaps surprisingly) difficult technical issue which arises in showing convergence to SLE_6. This is joint work with I. Binder (Toronto) and L. Chayes (UCLA).

1. **Critical Percolation** Consider some regular lattice structure, e.g., the bond square lattice or hexagonal tiling of $\mathbf{R}^2$. We let bonds, sites, or tiles be blue with probability p and yellow with probability $1-p$. It is well known that there is a critical value p_c such that if $p > p_c$ then there is a.s. an infinite blue cluster and for $p < p_c$ there is a.s. an infinite yellow cluster on the suitably defined dual lattice; at $p = p_c$ there is no percolation of either species. We note that the value of p_c is very lattice dependent; it happens to be $1/2$ for both bond percolation on the square lattice and hexagonal tiling (which is the same as site percolation on the triangular lattice).

2. **Scaling Limit** We would like to study critical percolation. Given a domain Ω, we tile it with a lattice at scale ε and perform percolation at the critical value. We would then like to study the limit as $\varepsilon \to 0$.

3. **Conformal Invariance** & **Cardy's Formula** A natural observable is crossing probability. Let Ω be a rectangle and look at the probability of a say left right blue crossing. What does this converge to as $\varepsilon \to 0$? Crossing probabilities are believed to be conformally invariant and given by Cardy's Formula in the limit. Cardy's Formula on the upper half plane takes the exact form shown. While crossing probabilities at criticality are supposed to converge to Cardy's Formula regardless of what lattice we start with, only a few models have been rigorously shown to satisfy Cardy's Formula: There is the celebrated result of S. Smirnov for hexagonal tiling. The proof there is based on the beautiful observation that complete color symmetry at the microscopic level leads to derivative relations between crossing probability type functions. In a previous joint work with L. Chayes, we were able to provide a modest generalization of Smirnov's result to a one–parameter family of triangular type models. This is all I will say on Cardy's Formula, so now let us return to the "big" picture.

4. **Interface** We can begin to under the scaling limit by understanding the interface: Let us impose boundary conditions on Ω by coloring half the boundary blue and the other half yellow, then given any percolation configuration there is an interface separating the blue cluster connected to the blue boundary from the yellow cluster connected to the yellow boundary. We can also think of this as a very simple process: We start at one corner and reveal say hexagons one by one by flipping a coin. Since everything is independent, these two descriptions are the same. In any case, an important thing to note is that we have the domain Markov property: The law of the interface conditioned on an initial portion is the same as the law of the interface if we started in the slit domain formed by the corresponding curve segment.

5. **Schramm's Principle** In principle, if the law for random curves is conformally invariant and satisfy the Markov property, then it must be SLE_κ for some κ.

6. **Discrete Markov Property** What we really have from Cardy's Formula is conformal invariance of the crossing probability, but we can put this together with the domain Markov property of the explorer process. Namely, let us run the explorer process until some time t, and ask first for the conditional probability of a left right blue crossing. A blue crossing either hits the explorer process or it doesn't, but in either case it forms a crossing of the corresponding slit domain, and vice versa. So we have the first equality. Integrating this by averaging over all possible initial segments of the process, we obtain the last display. What we will later use is actually a version of this equation with the process stopped at two times $s < t$.

7. **Smirnov's Framework** Smirnov in his ICM in 2006 outlined a framework for showing convergence to SLE_6 provided Cardy's Formula has been established. The first step is to invoke the result of Aizenman–Burchard that the law of the curves converge weakly in the sup–norm. Then *a priori* estimates are used to show that the limit is supported on "nice" (i.e., Löwner) curves. Next we take a limit of the discrete Markov property. The resulting expression is then rewritten using a random driving term $w(t)$ and Cardy's Formula. Finally, we expand at ∞ and will learn that $w(t)$ and $w(t)^2 - 6t$ are martingales, from which we conclude that we have SLE_6 (the expansion at ∞ idea was first used by Lawler–Schramm–Werner in the proof of the loop erased random walk to SLE_2). The second step requires some uniform continuity of the crossing probabilities and turned out to be particularly difficult. We will focus on this in the remainder of the talk.

8. **Discrete Markov Property Revisited** We begin by taking a much closer look at the equation expressing the discrete Markov property. We would like both sides of the equation to converge to the corresponding objects in the continuum. Notice that *a priori* there are actually three types of ε's: 1) There is the scale at which the explorer process is drawn; 2) the scale at which percolation is performed; and 3) the scale corresponding to the measure. In any case, taking the limit is non–trivial and we certainly need some statement of the uniform continuity of the relevant function, which is the crossing probability function. That is, we need at least some statement that if two curves are close in the sup–norm, then the crossing probabilities in the corresponding slit domains are also close. Something like this is indeed true: Outside a bad set with vanishingly small measure, the corresponding crossing probabilities can be made arbitrarily close by taking ε small enough.

9. **Critical Properties** This lemma will be proved by heavily using properties of critical percolation. The fundamental property of criticality is that there are scale–invariant bounds on crossing probabilities: There are upper and lower bounds for the crossing probability of a L by γL rectangle which only depend on γ (and not L). (This is true for both blue and yellow crossings.) There are also power law decay of correlations: The probability of a say blue connection between two points x and y is bounded by the distance between them divided by the lattice spacing raised to some power. In particular, the probability of a connection between two points separated by a positive distance goes to zero as $\varepsilon \to 0$.

10. **"Russo–Seymour–Welsh"** Perhaps the most widely used property is the so–called RSW estimates. The bounds on the previous slide can be "glued" together using correlation inequalities to derive scale invariant bounds on existence of blue (or yellow) rings in an annulus. Typically, we have some $\eta < \delta$ and set up $\log(\delta/\eta)$ annuli; the probability that there exists a ring in each annulus is uniformly bounded below by α, independently. Therefore the probability there does not exist a ring in any annulus tends to zero as $\eta/\delta \to 0$.

11. **Plausible** We can now see that the statement of the lemma is plausible. Suppose we have two curves γ_g and γ_r which are η–close in the sup–norm. In the picture here we have superimposed the two relevant slit domains; we perform percolation in the original domain and ask whether a left right blue crossing has been achieved for the slit domains. Suppose there is a blue crossing for the slit domain associated with γ_r but not γ_g; this necessarily means that all blue crossings for the red slit domain must land on the red curve. But then conditioning on the highest such crossing, we can set up many annuli, and with probability tending to one, the blue crossing can be continued to be a blue crossing for γ_g as well, and hence configurations with a crossing only for red and not green are highly unlikely.

12. **Topology of the Slit Domain** However, a careful examination of the topological situation involved reveals a problem. Note the boundary is colored half blue and half yellow, so the red and green curves are two–sided, with the convention that coming from a, blue is on the right. In the situations shown, it is somewhat easier to get a crossing for the domain on the left than the one on the right.

13. **"Counterexample"** This leads to the following "counterexample": Here the red curve is still close to the green curve, but the crossing probabilities are not close: The red curve has the blue side "exposed" whereas the green curve has the blue side on the "inside", so the crossing probability for red is higher than that for green, since basically any blue path which hits the red curve will form a valid crossing, whereas for a blue path to form a valid crossing by hitting the green curve is very difficult.

14. **Nodoublingback** However, the curves we consider are explorer path typical curves, that is, they are drawn from possible percolation interfaces. By RSW type estimates, we can show that the probability of a segment of a say blue path of diameter δ staying within a distance η of its previous self is exponentially small. By invoking this lemma at a larger scale than η, this will allow us to basically rule out for example the previous scenario. We also note that the estimate here only depends on the ratio δ/η.

15. **Reduction to Highest Crossing** In the remaining time I will try to indicate how to use the nodoublingback estimates. First let us reduce to the case of a blue crossing for the red domain and not the green and further to the case that the highest blue crossing for the red domain hits the red curve and does not intersect the green curve. (This is actually highly non–trivial.) So the blue crossing joins the red curve at some point M and the domain above this combined curve is conditioned on whereas the complement is unconditioned. Notice that the portion of the red curve after M is entirely contained in the unconditioned region. Our goal would be to use RSW to continue the blue crossing to the green curve in the unconditioned domain, but first we must guarantee that any such blue crossing must hit the blue side of the green curve (if it indeed hits the green curve).

16. **The Correct Topological Picture** The correct topological picture is as follows: Let us draw a large ball around the point where the blue crossing hits the red curve and suppose we can join the green curve to d in the complement of everything by Γ. Then the domain below Γ and the green curve forms a Jordan domain, with the external boundary blue. So now we are guaranteed to hit the *blue side* of the green curve if we can continue the blue crossing by RSW inside the ball to hit the green curve.

17. **The Point** $\gamma(t^*)$ Applying nodoublingback to say the green curve allows us to produce a point $\gamma(t^*)$ which happens much later than $\gamma(m)$ and which is very far away from the boundary of the relevant domain. In particular, the ball of radius δ around $\gamma(t^*)$ must all be contained in the unconditioned region.

18. **Where is** $\gamma(t^*)$**?** Recall what we are trying to do. We basically want the point $\gamma(t^*)$ to be in the connected component of d in the appropriate domain F_G which is formed by deleting from Ω a large neighborhood around M and the relevant portion of the green curve and the blue crossing. The domain F_G also has many smaller components, formed by say the green curve coming in and out of the neighborhood around M, but we can show by long–arm estimates that it is unlikely that $\gamma(t^*)$ is in these components. So we just need to rule out the possibility that $\gamma(t^*)$ is in the connected component of b. This cannot happen because we can compare the situation with the red version of this domain F_R, where we know that $\gamma(t^*)$ is in the correct connected component. This means the same must be true in F_G, since $\gamma(t^*)$ is very far away from all relevant boundaries of these domains.

19. **Problem with RSW** Now it remains to show that we can continue the blue crossing inside the big neighborhood around M to the green curve. Now we run into a somewhat subtler problem. We know that if the continuation hits the green curve then it hits on the correct side, but notice in the picture the green curve could "hide" behind the red curve and we could very well have a situation where the blue continuation fails to hit the green curve, since the blue ring is only "valid" in the unconditioned region.

20. **Multiscale Nodoublingback** In some sense this is the same sort of topological problem that was resolved by the point $\gamma(t^*)$. Our solution is to apply nodoublingback at many scales around M to produce a "good" point on a positive fraction of such scales. This can be done with high probability since our estimate for nodoublingback is scale invariant. Around each such "good" point we can then do a further RSW construction to guarantee that a blue continuation hits the green curve.