Strikwerda 4.4

4.4.1. *Determine if these polynomials are Schur or von Neumann polynomials, or neither.*

(a) $2z^3 + z^2 + z + 1$

$$\begin{aligned}
\varphi_3(z) &= 2z^3 + z^2 + z + 1 \\
\varphi_3^*(z) &= z^3 + z^2 + z + 2 \\
c_3 &= |\varphi_3^*(0)|^2 - |\varphi_3(0)|^2 \;=\; 3 \;>\; 0 \\
\psi(z) &= \frac{\varphi_3^*(0)\varphi_3(z) - \varphi_3(0)\varphi_3^*(z)}{z} \;=\; 3z^2 + z + 1 \;=\; \varphi_2(z) \\
\varphi_2^*(z) &= z^2 + z + 3 \\
c_2 &= |\varphi_2^*(0)|^2 - |\varphi_2(0)|^2 \;=\; 2 > 0 \\
\psi(z) &= \frac{\varphi_2^*(0)\varphi_2(z) - \varphi_2(0)\varphi_2^*(z)}{z} \;=\; 8z + 2 \;=\; \varphi_1(z) \\
\varphi_1^*(z) &= 2z + 8 \\
c_1 &= |\varphi_1^*(0)|^2 - |\varphi_1(0)|^2 \;=\; 6 > 0 \\
\psi(z) &= \frac{\varphi_1^*(0)\varphi_1(z) - \varphi_1(0)\varphi_1^*(z)}{z} \;=\; 60
\end{aligned}$$

Since $c_3, c_2, c_1 > 0$, the polynomial is a Schur polynomial.

(b) $2z^4 + z^3 + z^2 + 2z + 1$

$$\begin{aligned}
\varphi_4(z) &= 2z^4 + z^3 + z^2 + 2z + 1 \\
c_4 &= 1 \\
\varphi_3(z) &= 3z^3 + z^2 + 3 \\
c_3 &= 0 \\
\psi(z) &= 0z^2 - 3z + 3
\end{aligned}$$

Since the coefficient of degree $d - 1$ in $\psi(z)$ is zero, the polynomial is not a von Neumann polynomial (or a Schur polynomial). Solving analytically, two roots have radius $r \approx 1.03002$ from the origin.

(c) $z^6 + z^5 - z - 1$

Using MATLAB program:

```
>> polytype([1,1,0,0,0,-1,-1])

p6 = 1z^6+1z^5+0z^4+0z^3+0z^2-1z-1
c6 = 0
psi = 0z^5+0z^4+0z^3+0z^2+0z+0

p5 = 6z^5+5z^4+0z^3+0z^2+0z-1
c5 = 35
psi = 35z^4+30z^3+0z^2+0z+5

p4 = 35z^4+30z^3+0z^2+0z+5
c4 = 1200
psi = 1200z^3+1050z^2+0z-150

p3 = 1200z^3+1050z^2+0z-150
c3 = 1.4175e+06
psi = 1.4175e+06z^2+1.26e+06z+157500

p2 = 1.4175e+06z^2+1.26e+06z+157500
c2 = 1.9845e+12
psi = 1.9845e+12z+1.5876e+12

p1 = 1.9845e+12z+1.5876e+12
c1 = 1.41777e+24
psi = 1.41777e+24

Von Neumann polynomial, von Neumann order 1
```

```
function polytype(p)
%POLYTYPE  Determines if a polynomial is Schur or von Neumann

p = p(:).';
d = length(p) - 1;
c = zeros(1,d);
Order = 0;

while d > 0
        fprintf('\np%d = %s\n',d,poly2str(p));
        pd = conj(fliplr(p));                        % Step 1
        c(d) = abs(pd(d+1))^2 - abs(p(d+1))^2;      % Step 2
        fprintf('c%d = %g\n',d,c(d));
        psi = pd(d+1)*p - p(d+1)*pd;                 % Step 3
        psi = psi(1:d);
        fprintf('psi = %s\n',poly2str(psi));

        if all(psi == 0)                             % Step 4.1
                Order = Order + 1;
                p = (d:-1:1).*p(1:d);
        elseif psi(1) == 0                           % Step 4.2
                fprintf('\nNot a von Neumann polynomial\n');
                return;
        else                                         % Step 4.3
                p = psi;
        end

        d = d - 1;
end

if min(c) > 0
        fprintf('\nSchur polynomial, von Neumann order %d\n',Order);
elseif min(c) >= 0
        fprintf('\nVon Neumann polynomial, von Neumann order %d\n',Order);
else
        fprintf('\nNot a von Neumann polynomial (min c = %g)\n',min(c));
end

return;


function s = poly2str(p)
% Converts polynomials to strings
csign = '-++';
s = [sprintf('%gz^%d',p(1),length(p)-1),...
        sprintf('%+gz^%d',[p(2:length(p));(length(p)-2:-1:0)])];
s = strrep(strrep(strrep(s,'z^0',''),'^1+','+'),'^1-','-');
return;
```

Indeed, this is a von Neumann polynomial since $z^6 + z^5 - z - 1 = (z^5 - 1)(z + 1)$.

(d) $z^8 + z^7 + z^4 + z + 1$

```
>> polytype([1,1,0,0,1,0,0,1,1])

Von Neumann polynomial, von Neumann order 1
```

(e) $z^8 + z^5 + z + 1$

```
>> polytype([1,0,0,1,0,0,0,1,1])

Not a von Neumann polynomial
```

4.4.2. *Use the methods of this section to show that the leapfrog scheme* $\frac{v_m^{n+1}-v_m^{n-1}}{2k} + a\delta_0 v_m^n = 0$ *is stable iff* $|a\lambda| < 1$.

The amplification polynomial is

$$\Phi(z,\theta) \;=\; \tfrac{1}{2}(z^2-1) + \tfrac{ak}{2h}(2i\sin\theta)z \;=\; \tfrac{1}{2}z^2 + i(a\lambda\sin\theta)z - \tfrac{1}{2}.$$

Set $\varphi_2(z) = \frac{1}{2}z^2 + i\beta z - \frac{1}{2}$ with $\beta = a\lambda\sin\theta$. The scheme is stable iff φ_2 is a simple von Neumann polynomial. Since

$$\varphi_1(z) \;=\; \frac{\varphi_2^*(0)\varphi_2(z) - \varphi_2(0)\varphi_2^*(z)}{z} \;=\; \tfrac{1}{z}\left[\tfrac{1}{2}(\tfrac{1}{2}z^2 + i\beta z - \tfrac{1}{2}) + \tfrac{1}{2}(-\tfrac{1}{2}z^2 - i\beta z + \tfrac{1}{2})\right] \equiv 0,$$

by Theorem 4.3.2, φ_2 is a simple von Neumann iff $\varphi_2'(z) = z + i\beta$ is a Schur polynomial. Its root is $-i\beta$, so φ_2' is a Schur polynomial iff

$$|\beta| < 1 \quad \Longleftrightarrow \quad |a\lambda||\sin\theta| < 1 \;\forall\theta.$$

Therefore, the scheme is stable iff $|a\lambda| < 1$.

4.4.3. *Using the methods of this section, verify that the scheme*

$$\frac{3v_m^{n+1} - 4v_m^n + v_m^{n-1}}{2k} + a\delta_0 v_m^{n+1} = f_m^{n+1}$$

is unconditionally stable.

The amplification polynomial (given) is

$$\Phi(z,\theta) = \tfrac{1}{2}(3 + 2ia\lambda\sin\theta)z^2 - 2z + \tfrac{1}{2}.$$

Set $\varphi_2(z) = (\frac{3}{2} + i\beta)z^2 - 2z + \frac{1}{2}$ with $\beta = a\lambda\sin\theta$. Since $|\varphi_2^*(0)|^2 - |\varphi_2(0)|^2 = (\frac{9}{4} + \beta^2) - (\frac{1}{4}) > 0$, Theorem 4.3.2 implies that the scheme is stable if

$$\varphi_1(z) \;=\; \frac{\varphi_2^*(0)\varphi_2(z) - \varphi_2(0)\varphi_2^*(z)}{z} \;=\; (2+\beta^2)z + (-\tfrac{1}{2} + i\beta)$$

is a simple von Neumann polynomial. This is when

$$\begin{aligned} |-\tfrac{1}{2} + i\beta|^2 &\le |2+\beta^2|^2 \\ \Longleftrightarrow \quad \beta^2 + \tfrac{1}{4} &\le \beta^4 + 4\beta^2 + 4 \\ \Longleftrightarrow \quad 0 &\le \beta^4 + 3\beta^2 + \tfrac{15}{4}. \end{aligned}$$

But this is always true, so the scheme is unconditionally stable.

4.4.4. *Show that the modified leapfrog scheme*

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\delta_0 \left(\frac{v_m^{n+1} + 4v_m^n + v_m^{n-1}}{6} \right) = f_m^n$$

is stable iff $|a\lambda| < \sqrt{3}$.

The amplification polynomial is

$$\phi_2(z) \ = \ \tfrac{1}{2}(z^2 - 1) + \tfrac{ak}{6} \frac{2i \sin\theta}{2h}(z^2 + 4z + 1) \ = \ \overline{\beta} z^2 + 4i\alpha z - \beta.$$

Where $\alpha = \frac{a\lambda}{6} \sin\theta$ and $\beta = (\frac{1}{2} - i\alpha)$. Since

$$\begin{aligned} \varphi_1(z) \ &= \ \frac{\varphi_2^*(0)\varphi_2(z) - \varphi_2(0)\varphi_2^*(z)}{z} \\ &= \ \tfrac{1}{z} \left[\beta \left(\overline{\beta} z^2 + 4i\alpha z - \beta \right) + \beta \left(-\overline{\beta} z^2 - 4i\alpha z + \beta \right) \right] \equiv 0, \end{aligned}$$

Theorem 4.3.2 implies φ_2 is a simple von Neumann polynomial iff $\varphi_2'(z) = (1 + 2i\alpha)z + 4i\alpha$ is a Schur polynomial, which is iff

$$\begin{aligned} |4i\alpha|^2 \ &< \ |1 + 2i\alpha|^2 \\ \Longleftrightarrow \ \tfrac{1}{3}|a\lambda|^2 \sin^2\theta \ &< \ 1 \\ \Longleftrightarrow \ |a\lambda||\sin\theta| \ &< \ \sqrt{3}. \end{aligned}$$

So the scheme is stable iff $|a\lambda| < \sqrt{3}$.

4.4.5. *Find when the following explicit (4,4) scheme is stable.*

$$\frac{v_m^{n+2}-v_m^{n-2}}{4k}+a\left(1-\frac{h^2}{6}\delta^2\right)\delta_0\left(\frac{2v_m^{n+1}-v_m^n+2v_m^{n-1}}{3}\right)=\frac{2f_m^{n+1}-f_m^n+2f_m^{n-1}}{3}$$

The amplification polynomial is

$$\begin{aligned}\varphi_4(z) &= z^4-1+4ka\left(1-\tfrac{1}{6}(2\cos\theta-2)\right)\frac{2i\sin\theta}{2h}\left(\frac{2z^3-z^2+2z}{3}\right)\\ &= z^4+i\tfrac{4ak}{3h}\left(1-\tfrac{1}{3}\cos\theta+\tfrac{1}{3}\right)\sin\theta\left(2z^3-z^2+2z\right)-1\\ &= z^4+i\tfrac{4a\lambda}{9}\left(4-\cos\theta\right)\sin\theta\left(2z^3-z^2+2z\right)-1.\end{aligned}$$

Let $\beta=\frac{4a\lambda}{9}\left(4-\cos\theta\right)\sin\theta$ such that $\varphi_4(z)=z^4+i\beta(2z^3-z^2+2z)-1$. By (4.3.10), we have

$$\varphi_3(z) \;=\; \tfrac{1}{z}\left[(z^4+i\beta(2z^3-z^2+2z)-1)+(-z^4-i\beta(2z^3-z^2+2z)+1)\right]\equiv 0,$$

so by Theorem 4.3.2 the scheme is stable iff $\psi_3(z)=\frac{1}{2}\varphi_4'(z)=2z^3+i3\beta z^2-i\beta z+i\beta$ is a Schur polynomial. Using (4.3.10) and Theorem 4.3.2 again, ψ_3 is a Schur polynomial iff

$$\begin{aligned}\psi_2(z) &= \tfrac{1}{z}\left[2(2z^3+i3\beta z^2-i\beta z+i\beta)-i\beta(-i\beta z^3+i\beta z^2-i3\beta z+2)\right]\\ &= (4-\beta^2)z^2+(i6\beta+\beta^2)z+(-i2\beta-3\beta^2)\end{aligned}$$

is a Schur polynomial and $|\psi_3(0)|<|\psi_3^*(0)|$, or equivalently, $|\beta|<2$. Iterating once more, ψ_2 is a Schur polynomial iff

$$\begin{aligned}\psi_1(z) &= \tfrac{1}{z}\Big[(4-\beta^2)\left((4-\beta^2)z^2+(\beta^2+i6\beta)z+(-3\beta^2-i2\beta)\right)\\ &\qquad +(3\beta^2+i2\beta)\left((-3\beta^2+i2\beta)z^2+(\beta^2-i6\beta)z+(4-\beta^2)\right)\Big]\\ &= -4(2\beta^4+3\beta^2-4)z+2\beta(\beta^3-i11\beta^2+8\beta+i12)\end{aligned}$$

is a Schur polynomial and $|\psi_2(0)|<|\psi_2^*(0)|$, or equivalently,

$$9\beta^4+4\beta^2<(4-\beta^2)^2 \quad\Longleftrightarrow\quad |\beta|<\tfrac{1}{2}\sqrt{\sqrt{41}-3}<1.$$

Finally, ψ_1 is a Schur polynomial iff

$$|2\beta(\beta^3-i11\beta^2+8\beta+i12)|^2<|-4(2\beta^4+3\beta^2-4)|^2 \quad\Longleftrightarrow\quad |\beta|<\tfrac{1}{\sqrt{3}}.$$

To bound $(4-\cos\theta)\sin\theta$, find that its derivative is zero $-2\cos^2\theta+4\cos\theta+1=0$ at $\cos\theta=1-\frac{1}{2}\sqrt{6}$. This yields the bound $|(4-\cos\theta)\sin\theta|\le\sqrt{\frac{3}{2}}(1+\sqrt{6})(\sqrt{6}-\frac{3}{2})^{1/2}$. Therefore, the scheme is stable when

$$|a\lambda| \;<\; \frac{\frac{1}{\sqrt{3}}}{\frac{4}{9}\sqrt{\frac{3}{2}}(1+\sqrt{6})(\sqrt{6}-\frac{3}{2})^{1/2}} \;=\; \frac{3\sqrt{2}}{4}\frac{1}{(1+\sqrt{6})(\sqrt{6}-\frac{3}{2})^{1/2}}.$$

4.4.6. *Show that the following scheme for $u_t + au_x = f$ is accurate of order (3,4) and is unstable for all values of λ:*

$$\frac{11v_m^{n+1} - 18v_m^n + 9v_m^{n-1} - 2v_m^{n-2}}{6k} + a\left(1 + \frac{h^2}{6}\delta^2\right)^{-1}\delta_0 v_m^{n+1} = f_m^{n+1}.$$

The symbols are

$$\begin{aligned}
p_{h,k}(s,\xi) &= \left(\frac{2+\cos\xi h}{3}\right)\frac{11\mathrm{e}^{sk} - 18 + 9\mathrm{e}^{-sk} - 2\mathrm{e}^{-2sk}}{6k} + a\frac{i\sin\xi h}{h}\mathrm{e}^{sk} \\
&= (s + ia\xi)(1 + sk + \tfrac{1}{2}s^2k^2 - \tfrac{1}{6}\xi^2h^2 - \tfrac{1}{6}s\xi^2kh^2 - \tfrac{1}{12}s^2\xi^2k^2h^2) + O(k^3) + O(h^4) \\
r_{h,k}(s,\xi) &= \frac{2+\cos\xi h}{3}\mathrm{e}^{sk} \;=\; \tfrac{1}{3}(2 + 1 - \tfrac{1}{2}\xi^2h^2)(1 + sk + \tfrac{1}{2}s^2k^2) + O(k^3) + O(h^4) \\
&= 1 + sk + \tfrac{1}{2}s^2k^2 - \tfrac{1}{6}\xi^2h^2 - \tfrac{1}{6}s\xi^2kh^2 - \tfrac{1}{12}s^2\xi^2k^2h^2 + O(k^3) + O(h^4),
\end{aligned}$$

and $r_{h,k}(s,\xi)(s + ia\xi) = p_{h,k}(s,\xi) + O(k^3) + O(h^4)$. Therefore the scheme is (3,4) accurate.

The amplification polynomial is

$$\begin{aligned}
\varphi_3(z) &= (11z^3 - 18z^2 + 9z - 2) + 6ka\left(\frac{3}{2+\cos\theta}\right)\frac{i\sin\theta}{h}z^3 \\
&= (11z^3 - 18z^2 + 9z - 2) + i18a\lambda\frac{\sin\theta}{2+\cos\theta}z^3 \\
&= (11 + i\beta)z^3 - 18z^2 + 9z - 2
\end{aligned}$$

where $\beta = 18a\lambda\frac{\sin\theta}{2+\cos\theta}$.