Strikwerda 4.1

4.1.1. *Show that the implicit (2,4) leapfrog scheme*

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\left(1 + \frac{h^2}{6}\delta^2\right)^{-1} \delta_0 v_m^n = f_m^n$$

for the one-way wave equation with λ constant is stable if and only if $|a\lambda| < \frac{1}{\sqrt{3}}$.

Substituting $g^{n'} \mathrm{e}^{im'\theta}$ for $v_{m'}^{n'}$,

$$\begin{aligned}
0 &= \tfrac{1}{2k}(g - g^{-1}) + a(1 + \tfrac{1}{3}(\cos\theta - 1))^{-1} \tfrac{i}{h} \sin\theta \\
0 &= \tfrac{1}{2}g^2 + i\frac{3a\lambda\sin\theta}{2+\cos\theta}g - \tfrac{1}{2} \\
g(\theta) &= -i\frac{3a\lambda\sin\theta}{2+\cos\theta} \pm \left[1 - \left(\frac{3a\lambda\sin\theta}{2+\cos\theta}\right)^2\right]^{1/2}.
\end{aligned}$$

The argument of the square root is positive when $|a\lambda| < \frac{1}{\sqrt{3}}$, since then

$$(a\lambda)^2 < \tfrac{1}{3} \le \left(\frac{2+\cos\theta}{3\sin\theta}\right)^2 \quad \Rightarrow \quad \left(\frac{3a\lambda\sin\theta}{2+\cos\theta}\right)^2 \le 1.$$

But if $|a\lambda| = \frac{1}{\sqrt{3}}$, then $g_+ = g_-$ at $\theta = \frac{2}{3}\pi$. The amplification factor for $|a\lambda| \le \frac{1}{\sqrt{3}}$ is

$$|g(\theta)|^2 = 1 - \left(\frac{3a\lambda\sin\theta}{2+\cos\theta}\right)^2 + \left(\frac{3a\lambda\sin\theta}{2+\cos\theta}\right)^2 = 1.$$

This requires that $|a\lambda| < \frac{1}{\sqrt{3}}$ are never equal, otherwise there are solutions with near linear growth. Therefore the scheme is stable iff $|a\lambda| < \frac{1}{\sqrt{3}}$.

4.1.2. *Show that the (2,2) leapfrog scheme for $u_t + au_{xxx} = f$ (see (2.2.15)) given by*

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\delta^2\delta_0 v_m^n = f_m^n,$$

with $\nu = k/h^3$ constant, is stable iff $|a\nu| < \frac{2}{3^{3/2}}$.

Substituting $g^{n'} \mathrm{e}^{im'\theta}$ for $v_{m'}^{n'}$,

$$\begin{aligned}
0 &= \frac{g - g^{-1}}{2k} + a\frac{2\cos\theta - 2}{h^2}\frac{2i\sin\theta}{2h} \\
0 &= \tfrac{1}{2}g - i(4a\nu\sin^2\tfrac{1}{2}\theta\sin\theta)g - \tfrac{1}{2} \\
g(\theta) &= i(4a\nu\sin^2\tfrac{1}{2}\theta\sin\theta) \pm \left[1 - (4a\nu\sin^2\tfrac{1}{2}\theta\sin\theta)^2\right]^{1/2}.
\end{aligned}$$

Since $(4a\nu\sin^2\frac{1}{2}\theta\sin\theta)^2$ has maximum value $\frac{27}{64}$ (at $\theta = \frac{2}{3}\pi$), we have $g_+(\theta) \ne g_-(\theta)$ for all θ iff $|a\nu| < \frac{1}{4}\sqrt{\frac{64}{27}} = \frac{2}{3^{3/2}}$. As in the previous problem, $|g(\theta)|^2 \equiv 1$ when $g_+ \ne g_-$.

4.1.3. *Show that the leapfrog scheme*

$$\frac{v_m^{n+1} - v_m^{n-1}}{2k} + a\left(1 - \frac{h^2}{6}\delta^2 + \frac{h^4}{30}\delta^4\right)\delta_0 v_m^n = f_m^n$$

for the one-way wave equation is accurate of order (2,6) and, if λ *is constant, is stable iff*

$$|a\lambda| < \frac{3}{[2(\frac{2}{5})^{1/3} - 1]^{1/2}\,[(\frac{5}{2})^{2/3} + 3(\frac{5}{2})^{1/3} + 1]}.$$

Finding the amplification factor:

$$0 \quad = \quad \frac{g - g^{-1}}{2k} + a\left(1 - \frac{2\cos\theta - 2}{6} + \frac{(2\cos\theta - 2)^2}{30}\right)\frac{2i\sin\theta}{2h}$$