Strikwerda 3.2

3.2.1. *Show that the (forward-backward) MacCormack scheme*

$$\begin{aligned}\tilde{v}_m^{n+1} &= v_m^n - a\lambda(v_{m+1}^n - v_m^n) + kf_m^n,\\ v_m^{n+1} &= \tfrac{1}{2}(v_m^n + \tilde{v}_m^{n+1} - a\lambda(\tilde{v}_m^{n+1} - \tilde{v}_{m-1}^{n+1}) + kf_m^{n+1})\end{aligned}$$

is a second-order accurate scheme for the one-way wave equation. Show that for $f = 0$ it is identical to the Lax-Wendroff scheme.

Rearranging, the scheme is

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} - \frac{a^2k}{2}\frac{v_{m+1}^n - 2v_m^n + v_{m-1}^n}{h^2} = \frac{1}{2}\left(f_m^n + f_m^{n+1} - ak\frac{f_m^n - f_{m-1}^n}{h}\right),$$

which is the same as the Lax-Wendroff scheme (3.1.2) if $f = 0$. The symbols are

$$\begin{aligned}p_{k,h}(s,\xi) &= \tfrac{1}{k}(\mathrm{e}^{sk} - 1) + \tfrac{a}{2h}(2i\sin\xi h) - \tfrac{a^2k}{2h^2}(2\cos\xi h - 2)\\ &= \tfrac{1}{k}(sk + \tfrac{1}{2}s^2k^2) + \tfrac{ia}{h}(\xi h) - a^2\tfrac{k}{h^2}(-\tfrac{1}{2}\xi^2h^2) + O(h^2) + O(k^2)\\ &= s + \tfrac{1}{2}s^2k + ia\xi + \tfrac{a^2}{2}\xi^2k + O(h^2) + O(k^2),\\ r_{k,h}(s,\xi) &= \tfrac{1}{2} + \tfrac{1}{2}\mathrm{e}^{sk} - \tfrac{ak}{2h} + \tfrac{ak}{2h}\mathrm{e}^{-i\xi h}\\ &= \tfrac{1}{2} + \tfrac{1}{2}(1 + sk) - \tfrac{ak}{2h} + \tfrac{ak}{2h}(1 - i\xi h) + O(h^2) + O(k^2)\\ &= 1 + \tfrac{1}{2}sk - \tfrac{ia}{2}\xi k + O(h^2) + O(k^2).\end{aligned}$$

Since

$$\begin{aligned}r_{h,k}(s,\xi)p(s,\xi) &= (1 + \tfrac{1}{2}sk - \tfrac{ia}{2}\xi k)(s + ia\xi) + O(h^2) + O(k^2)\\ &= s + \tfrac{1}{2}s^2k - \tfrac{ia}{2}s\xi k + ia\xi + \tfrac{ia}{2}s\xi k + \tfrac{a^2}{2}\xi^2k + O(h^2) + O(k^2)\\ &= p_{h,k}(s,\xi) + O(h^2) + O(k^2),\end{aligned}$$

the scheme is (2,2)-accurate.

3.2.2. *Show that the backward-time central-space scheme*

$$\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = 0$$

is unconditionally stable.

Rearranging, $v_m^{n+1} = v_m^n - \frac{ak}{2h}(v_{m+1}^{n+1} - v_{m-1}^{n+1})$. Substitute $g^{n'}\mathrm{e}^{im'\theta}$ for $v_{m'}^{n'}$ and then cancel the factor of $g^n\mathrm{e}^{im\theta}$ to obtain

$$\begin{aligned}g &= 1 - \tfrac{ak}{2h}(g\mathrm{e}^{i\theta} - g\mathrm{e}^{-i\theta})\\ g &= (1 + \tfrac{iak}{h}\sin\theta)^{-1}.\end{aligned}$$

Since $|g(\theta)|^2 = (1 + \frac{a^2k^2}{h^2}\sin^2\theta)^{-1} \leq 1$ for all θ regardless of a, k, and h, the scheme is unconditionally stable.

3.2.3. *Show that the box scheme*

$$\tfrac{1}{2k}\left[(v_m^{n+1}+v_{m+1}^{n+1})-(v_m^n+v_{m+1}^n)\right]+\tfrac{a}{2h}\left[(v_{m+1}^{n+1}-v_m^{n+1})+(v_{m+1}^n-v_m^n)\right]$$
$$=\tfrac{1}{4}(f_{m+1}^{n+1}+f_m^{n+1}+f_{m+1}^n+f_m^n)$$

is an approximation to the one-way wave equation that is accurate of order (2,2) and is stable for all values of λ.

The symbols for the scheme are

$$\begin{aligned}
p_{h,k}(s,\xi) &= \tfrac{1}{2k}(\mathrm{e}^{sk}-1)(\mathrm{e}^{i\xi h}+1)+\tfrac{a}{2h}(\mathrm{e}^{sk}+1)(\mathrm{e}^{i\xi h}-1)\\
&= \tfrac{1}{2k}(sk+\tfrac{1}{2}s^2k^2)(2+i\xi h)+\tfrac{a}{2h}(2+sk)(i\xi h-\tfrac{1}{2}\xi^2h^2)+O(k^2)+O(h^2)\\
&= s+\tfrac{1}{2}s^2k+\tfrac{i}{2}s\xi h+\tfrac{i}{4}s^2\xi kh+ia\xi+\tfrac{ia}{2}s\xi k-\tfrac{a}{2}\xi^2h-\tfrac{a}{4}s\xi^2kh+O(k^2)+O(h^2),\\
r_{h,k}(s,\xi) &= \tfrac{1}{4}(\mathrm{e}^{sk}+1)(\mathrm{e}^{i\xi h}+1)\\
&= \tfrac{1}{4}(2+sk)(2+i\xi h)+O(k^2)+O(h^2).
\end{aligned}$$

Since $r_{h,k}(s,\xi)p(s,\xi)=(1+\frac{1}{2}sk+\frac{i}{2}\xi h+\frac{i}{4}s\xi kh)(s+ia\xi)+O(k^2)+O(h^2)=p(s,\xi)+O(k^2)+O(h^2)$, the scheme is (2,2) accurate.

The amplifying factor is one:

$$\begin{aligned}
0 &= \tfrac{1}{2k}(g-1)(\mathrm{e}^{i\theta}+1)+\tfrac{a}{2h}(g+1)(\mathrm{e}^{i\theta}-1)\\
g(\theta) &= \frac{h(\mathrm{e}^{i\theta}+1)-ak(\mathrm{e}^{i\theta}-1)}{h(\mathrm{e}^{i\theta}+1)+ak(\mathrm{e}^{i\theta}-1)}\\
|g(\theta)|^2 &= \frac{[(h+ak)+(h-ak)\cos\theta]^2+(h-ak)^2\sin^2\theta}{[(h-ak)+(h+ak)\cos\theta]^2+(h+ak)^2\sin^2\theta}\\
&= \frac{(h+ak)^2+2(h^2-a^2k^2)\cos\theta+(h-ak)^2}{(h-ak)^2+2(h^2-a^2k^2)\cos\theta+(h+ak)^2} = 1.
\end{aligned}$$

So the scheme is stable for all values of λ.