Strikwerda 3.1

3.1.1. *Using equations* $\sum \alpha_\ell = 1$*,* $\sum \alpha_\ell \ell = -a\lambda$*, and* $\sum \alpha_\ell \ell^2 = a^2\lambda^2$*, show that the Lax-Wendroff scheme is the only explicit one-step second-order accurate scheme of the form* $v_m^{n+1} = \alpha_1 v_{m+1}^n + \alpha_0 v_m^n + \alpha_{-1} v_{m-1}^n$.

Any explicit one-step second-order accurate scheme must satisfy $\sum \alpha_\ell = 1$, $\sum \alpha_\ell \ell = -a\lambda$, and $\sum \alpha_\ell \ell^2 = a^2\lambda^2$. With the requirement that $\alpha_\ell = 0$ for $|\ell| > 1$, these equation reduce to

$$\left\{ \begin{array}{rcl} \alpha_1 + \alpha_0 + \alpha_{-1} & = & 1 \\ \alpha_1 - \alpha_{-1} & = & -a\lambda \\ \alpha_1 + \alpha_{-1} & = & a^2\lambda^2, \end{array} \right.$$

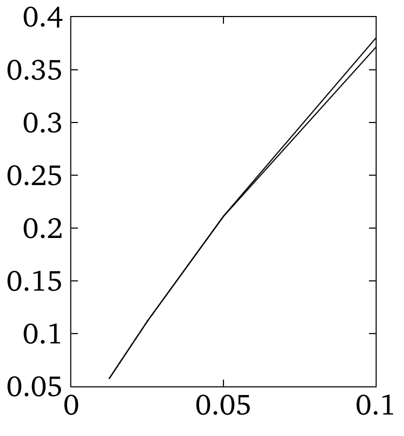
which has the unique solution $\alpha_1 = \frac{a\lambda}{2}(a\lambda - 1)$, $\alpha_0 = 1 - a^2\lambda^2$, $\alpha_{-1} = \frac{a\lambda}{2}(a\lambda + 1)$. To see that this is the Lax-Wendroff scheme, rearrange equation (3.1.1) to find

$$\begin{array}{rcl} v_m^{n+1} & = & v_m^n - \dfrac{a\lambda}{2}(v_{m+1}^n - v_{m-1}^n) + \dfrac{a^2\lambda^2}{2}(v_{m+1}^n - 2v_m^n + v_{m-1}^n) \\ & = & \underbrace{\tfrac{a\lambda}{2}(a\lambda - 1)}_{\alpha_1} v_{m+1}^n + \underbrace{(1 - a^2\lambda^2)}_{\alpha_0} v_m^n + \underbrace{\tfrac{a\lambda}{2}(1 + a\lambda)}_{\alpha_{-1}} v_{m-1}^n. \end{array}$$

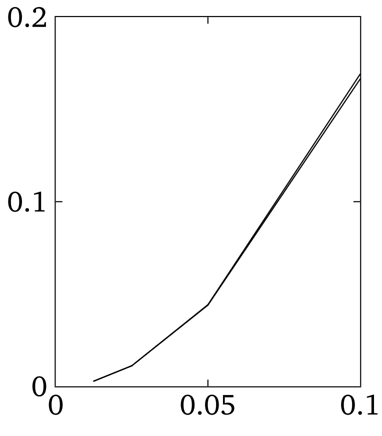
3.1.2. *Solve* $u_t + u_x = 0$*,* $-1 \le x \le 1$*,* $0 \le t \le 1.2$ *with* $u(0, x) = \sin 2\pi x$ *and periodicity, i.e.,* $u(t, 1) = u(t, -1)$*. Use the forward-time backward-space and Lax-Wendroff methods with* $\lambda = 0.8$*. Demonstrate the order of accuracy of each method using* $h = \frac{1}{10}$*,* $\frac{1}{20}$*,* $\frac{1}{40}$*, and* $\frac{1}{80}$*. Measure the error in the* L^2 *norm and the maximum norm.*

The plots below show the errors for the two methods vs. h (for each h, the two norms compute approximately the same errors). Indeed, the forward-time backward-space method is first-order accurate and the Lax-Wendroff method is second-order accurate.

Forward-Time Backward-Space Error



Lax-Wendroff Error



Implementation (MATLAB):

```
%%% Forward-time backward-space %%%
hset = [1/10,1/20,1/40,1/80];
L = 0.8;
L2Error = [];
MaxError = [];

for h = hset
    x = (-1/h:1:1/h-1)*h;
    v = sin(2*pi*x);
    t = 0;

    while t < 1.2
        v = filter([1-L,L],1,v,L*v(end));
        t = t + L*h;
    end

    vexact = sin(2*pi*(x-t));
    L2Error = [L2Error,sqrt(h)*norm(v - vexact,2)];
    MaxError = [MaxError,norm(v - vexact,inf)];
end

plot(hset,[L2Error;MaxError],'.-');
title('Forward-Time Backward-Space Error');
```

```
%%% Lax-Wendroff %%%
hset = [1/10,1/20,1/40,1/80];
L = 0.8;
L2Error = [];
MaxError = [];

for h = hset
    x = (-1/h:1:1/h-1)*h;
    v = sin(2*pi*x);
    t = 0;
    a = [0.5*L*(L - 1), 1 - L^2, 0.5*L*(1 + L)];

    while t < 1.2
        v = filter(a,1,v([2:end,1]),...
            [a(2)*v(1) + a(3)*v(end),a(3)*v(1)]);
        t = t + L*h;
    end

    vexact = sin(2*pi*(x-t));
    L2Error = [L2Error,sqrt(h)*norm(v - vexact,2)];
    MaxError = [MaxError,norm(v - vexact,inf)];
end

plot(hset,[L2Error;MaxError],'.-');
title('Lax-Wendroff Error');
```

3.1.3. *Solve the equation* $u_t + u_x = -\sin^2 u$ *with the scheme*

$$v_m^{n+1} = v_m^n - \frac{a\lambda}{2}(v_{m+1}^n - v_{m-1}^n) + \frac{a^2\lambda^2}{2}(v_{m+1}^n - 2v_m^n + v_{m-1}^n) + kf_m^n,$$

treating the $-\sin^2 u$ *term as* $f(t,x)$. *Show that the scheme is first-order accurate. The exact solution is* $u(t,x) = \cot^{-1}(\cot[u_0(x-t)] + t)$ *(Exercise 1.1.5). Use a smooth function, such as* $\sin(x-t)$, *as initial data and boundary data.*

Similar to the previous problem, this program tests the scheme with various step sizes. The plot verifies that the scheme is indeed first-order accurate.

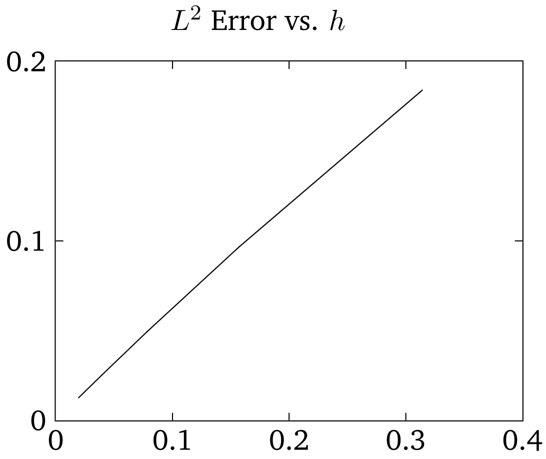
```
hset = 2*pi*[1/20,1/40,1/80,1/160,1/320];
L = 0.8;
L2Error = [];

for h = hset
    x = -pi:h:pi-h;
    v = sin(x);
    t = 0;
    a = [0.5*L*(L - 1), 1 - L^2, 0.5*L*(1 + L)];

    while t < 0.5
        v = filter(a,1,v([2:end,1]),...
            [a(2)*v(1) + a(3)*v(end),a(3)*v(1)]) ...
           - (L*h)*sin(v).^2;
        t = t + L*h;
    end

    vexact = atan(tan(sin(x-t))./(1 + t*tan(sin(x-t))));
    L2Error = [L2Error,sqrt(h)*norm(v - vexact,2)];
end

plot(hset,L2Error,'.-');
```



3.1.4. *Modify the scheme of Exercise 3.1.3 to be second-order accurate and explicit.*

Using $f_t = \frac{\mathrm{d}}{\mathrm{d}t}(-\sin^2 u) = -2\sin u \cos u = -\sin 2u$, we can replace the f_t term in the derivation of the Lax-Wendroff scheme to obtain

$$v_m^{n+1} = v_m^n - \tfrac{\lambda}{2}(v_{m+1}^n - v_{m-1}^n) + \tfrac{\lambda^2}{2}(v_{m+1}^n - 2v_m^n + v_{m-1}^n) - \tfrac{k^2}{2}\sin 2v_m^n - \tfrac{\lambda}{4}(f_{m+1}^n - f_{m-1}^n).$$

3.1.5. *Determine the order of accuracy of the Euler backward scheme in Exercise 2.2.6.*

Rearranging, the scheme becomes

$$\begin{aligned}
\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^{n+1/2} - v_{m-1}^{n+1/2}}{2h} &= f_m^{n+1} \\
\frac{v_m^{n+1} - v_m^n}{k} + a\frac{v_{m+1}^n - v_{m-1}^n}{2h} - a^2k\frac{v_{m+2}^n - 2v_m^n + v_{m-2}^n}{(2h)^2} &= f_m^{n+1} - ak\frac{f_{m+1}^n - f_{m-1}^n}{2h},
\end{aligned}$$

so its symbols are

$$\begin{aligned}
p_{k,h}(s,\xi) &= \tfrac{1}{k}(\mathrm{e}^{sk} - 1) + \tfrac{ia}{h}\sin\xi h - \tfrac{a^2k}{2h^2}(\cos 2\xi h - 1), \\
r_{k,h}(s,\xi) &= \mathrm{e}^{sk} - \tfrac{iak}{h}\sin\xi h.
\end{aligned}$$

The symbol of the differential operator $P = \frac{\mathrm{d}}{\mathrm{d}t} + a\frac{\mathrm{d}}{\mathrm{d}x}$ is $p(s,\xi) = s + ia\xi$. Applying Theorem 3.1.1, we evaluate the accuracy:

$$\begin{aligned}
&p_{k,h}(s,\xi) - r_{k,h}(s,\xi)p(s,\xi) \\
&\quad= \tfrac{1}{k}(\mathrm{e}^{sk} - 1) + \tfrac{ia}{h}\sin\xi h - \tfrac{a^2k}{2h^2}(\cos 2\xi h - 1) - (\mathrm{e}^{sk} - \tfrac{iak}{h}\sin\xi h)(s + ia\xi) \\
&\quad= \tfrac{1}{k}(1 + sk + \tfrac{1}{2}s^2k^2 - 1) + \tfrac{ia}{h}(\xi h) - \tfrac{a^2k}{2h^2}(1 - 2\xi^2h^2 - 1) \\
&\qquad\quad -s\mathrm{e}^{sk} - ia\xi\mathrm{e}^{sk} + \tfrac{iak}{h}s\sin\xi h + ia\tfrac{iak}{h}\xi\sin\xi h + O(k^2) + O(h^2) \\
&\quad= s + \tfrac{1}{2}s^2k + ia\xi + a^2\xi^2k \\
&\qquad\quad -s(1 + sk) - ia\xi(1 + sk) + \tfrac{iak}{h}s(\xi h) - \tfrac{a^2k}{h}\xi(\xi h) + O(k^2) + O(h^2) \\
&\quad= -\tfrac{1}{2}s^2k + O(k^2) + O(h^2).
\end{aligned}$$

So the scheme is accurate of order (1,2).

3.1.6. *Find the symbol and accuracy of the scheme*

$$v_m^{n+1} = \tfrac{1}{2}(v_{m+1}^n + v_{m-1}^n) - \tfrac{ak}{2h^3}(v_{m+2}^n - 2v_{m+1}^n + 2v_{m-1}^n - v_{m-2}^n) + kf_m^n.$$

We must first normalize the scheme such that $R_{k,h}1 = 1$:

$$\frac{v_m^{n+1} - \frac{1}{2}(v_{m+1}^n + v_{m-1}^n)}{k} + a\frac{v_{m+2}^n - 2v_{m+1}^n + 2v_{m-1}^n - v_{m-2}^n}{2h^3} \quad = \quad f_m^n.$$

The symbol of the scheme is thus

$$p_{k,h}(s,\xi) \quad = \quad \frac{\mathrm{e}^{sk} - \cos h\xi}{k} + ai\frac{\sin 2h\xi - 2\sin h\xi}{h^3}.$$

Since

$$\begin{aligned} \sin 2h\xi - 2\sin h\xi \;=\; 2\sin h\xi\cos h\xi - 2\sin h\xi \quad &= \quad 2(\cos h\xi - 1)\sin h\xi \\ &= \quad 2(-2\sin^2\tfrac{1}{2}h\xi)\sin h\xi, \end{aligned}$$

the symbol is equivalently

$$p_{k,h}(s,\xi) \;=\; \frac{\mathrm{e}^{sk} - \cos h\xi}{k} - 4ai\frac{\sin^2\frac{1}{2}h\xi\sin h\xi}{h^3}$$

(the problem statement has a sign error). The accuracy is

$$\begin{aligned} &p_{k,h}(s,\xi) - r_{k,h}(s,\xi)p(s,\xi) \\ &\quad= \quad \tfrac{1}{k}\mathrm{e}^{sk} - \tfrac{1}{k}\cos h\xi + \tfrac{ia}{h^3}(\sin 2h\xi - 2\sin h\xi) - (s - ia\xi^3) \\ &\quad= \quad \tfrac{1}{k}(1 + sk + \tfrac{1}{2}s^2k^2) - \tfrac{1}{k}(1 - \tfrac{1}{2}h^2\xi^2) + \tfrac{ia}{h^3}(2h\xi - \tfrac{8}{6}h^3\xi^3 - 2h\xi + \tfrac{2}{6}h^3\xi^3) \\ &\qquad\qquad -s + ia\xi^3 + O(k^2) + O(h^2) \\ &\quad= \quad \tfrac{1}{2}s^2k + \tfrac{1}{2}\xi^2\tfrac{h^2}{k} + O(k^2) + O(h^2) \end{aligned}$$

With $k = h$, the scheme is accurate of order 1.