

Convergence and Stability for Multistep Methods

We have been studying numerical methods to solve the initial value problem

$$\begin{cases} y'(t) = f(t, y(t)), & a \leq t \leq b \\ y(a) = \alpha \end{cases} \quad (1)$$

Let $h > 0$ and define $t_i = a + ih$. We want to find values w_i approximate that solution $y_i = y(t_i)$ for each i . Consider the $p + 1$ step multistep method

$$\begin{cases} w_0 = \alpha_0, \dots, w_p = \alpha_p, \\ w_{i+1} = \sum_{k=0}^p a_k w_{i-k} + h \sum_{k=-1}^p b_k f(t_{i-k}, w_{i-k}) \end{cases} \quad (2)$$

where a_0, \dots, a_p and b_{-1}, \dots, b_p are constants. We want to know when a method will work well. A method is said to be **convergent** if it can achieve arbitrary accuracy with a sufficiently small stepsize h ,

$$\lim_{h \downarrow 0} \max_{1 \leq i \leq N} |w_i - y(t_i)| = 0.$$

Important fact:

A consistent method is convergent if and only if it is stable.

Consistency

A method (2) is **consistent** if $\lim_{h \downarrow 0} \alpha_i = y_i$ for $i = 1, \dots, p$ and if the local truncation errors

$$\tau_{i+1}(h) = \frac{1}{h} \left[y_{i+1} - \sum_{k=0}^p a_k y_{i-k} - h \sum_{k=-1}^p b_k f(t_{i-k}, y_{i-k}) \right]$$

converge to zero as $h \downarrow 0$,

$$\lim_{h \downarrow 0} \max_{i > p} |\tau_i(h)| = 0. \quad (3)$$

For multistep methods, there is an easy check for consistency: (3) holds if and only if

$$\sum_{k=0}^p a_k = 1, \quad - \sum_{k=0}^p k a_k + \sum_{k=-1}^p b_k = 1.$$

Stability

On every step, a consistent method incurs a small error. A method is said to be **stable** if the error propagates without growing too fast. Consider applying (2) to the perturbed problem

$$\begin{cases} y'(t) = f(t, y(t)), & a \leq t \leq b \\ y(a) = \alpha + \epsilon & \epsilon \text{ is small} \end{cases}$$

Then (w_i) should not vary too much from the approximation for the unperturbed problem.

Using Taylor expansion about (t_0, y_0) , we can approximate

$$\begin{aligned} y'(t) &= f(t, y(t)) \\ &\approx f(t_0, y_0) + f_t(t_0, y_0)(t - t_0) + f_y(t_0, y_0)(y(t) - y_0) \\ \tilde{y}'(t) &= \underbrace{f_y(t_0, y_0)}_{\lambda} \tilde{y}(t) + \underbrace{f(t_0, y_0) + f_t(t_0, y_0)(t - t_0)}_{g(t)} \quad \tilde{y}(t) \approx y(t) - y_0 \\ \tilde{y}'(t) &= \lambda \tilde{y}(t) + g(t) \end{aligned}$$

for small $t - t_0$. So we can restrict our analysis of stability to equations of the form $y'(t) = \lambda y(t) + g(t)$.

In fact, we can restrict even further. Let $\epsilon_0, \dots, \epsilon_p$ be a small perturbations, and let

$$\begin{cases} w_0 = \alpha_0, \dots, w_p = \alpha_p, \\ w_{i+1} = \sum_{k=0}^p a_k w_{i-k} + h \sum_{k=-1}^p b_k f(t_{i-k}, w_{i-k}) \end{cases} \quad \begin{cases} v_0 = \alpha_0 + \epsilon_0, \dots, v_p = \alpha_p + \epsilon_p, \\ v_{i+1} = \sum_{k=0}^p a_k v_{i-k} + h \sum_{k=-1}^p b_k f(t_{i-k}, v_{i-k}) \end{cases}$$

where $f(t, y) = \lambda y + g(t)$. The difference due to the perturbation is

$$\begin{aligned} v_{i+1} - w_{i+1} &= \sum_{k=0}^p a_k (v_{i-k} - w_{i-k}) + h \sum_{k=-1}^p b_k [(\lambda v_{i-k} + g(t_{i-k})) - (\lambda w_{i-k} + g(t_{i-k}))] \\ &= \sum_{k=0}^p a_k (v_{i-k} - w_{i-k}) + h \sum_{k=-1}^p b_k \lambda (v_{i-k} - w_{i-k}), \end{aligned}$$

so $g(t)$ does not affect the stability analysis. Therefore, we can restrict to considering the homogeneous problem $y'(t) = \lambda y(t)$.

We determine stability by behavior on the **model problem**: $y'(t) = \lambda y(t)$, $\lambda \in \mathbb{C}$.

Stability of the Model Problem

Applying (2) to the model problem, we have

$$w_{i+1} = \sum_{k=0}^p a_k w_{i-k} + h\lambda \sum_{k=-1}^p b_k w_{i-k}$$

$$(1 - h\lambda b_{-1})w_{i+1} - \sum_{k=0}^p (a_k + h\lambda b_k)w_{i-k} = 0.$$

This is a homogeneous linear difference equation—the solution of such equations is analogous to homogeneous linear differential equations. Considering solutions of the form $w_i = r^i$, we obtain the **characteristic equation**,

$$(1 - h\lambda b_{-1})r^{p+1} - \sum_{k=0}^p (a_k + h\lambda b_k)r^{p-k} = 0. \quad (4)$$

Let $r_0(h\lambda), \dots, r_p(h\lambda)$ denote the **characteristic roots** of this equation. We shall see that these roots determine the stability. If all roots have multiplicity one, the general solution is

$$w_i = \sum_{k=0}^p c_k r_k(h\lambda)^i$$

where the c_k are determined by the initial conditions (and if $r_k(\lambda h)$ is a multiple root, there are solutions $r_k(\lambda h)^n, nr_k(\lambda h)^n, n^2 r_k(\lambda h)^n, \dots$).

When $\lambda h = 0$, the characteristic equation is

$$r^{p+1} - \sum_{k=0}^p a_k r^{p-k} = 0.$$

Consider the problem $y' \equiv 0$ (so $\lambda = 0$) and the initial values $w_0 = w_1 = \dots = w_p = 0$, then of course $w_i = 0$ for all i . Compare this to the approximation with perturbed initial values

$$z_0 = \epsilon, z_1 = \epsilon r_k(0), \dots, z_p = \epsilon r_k(0)^p.$$

Then $z_i = \epsilon r_k(0)^i$, and

$$|w_i - z_i| = \epsilon |r_k(0)|^i.$$

So the method is unstable if for any root $|r_k(0)| > 1$. Conversely, if $|r_k(0)| \leq 1$, then the error $|w_i - z_i|$ is bounded. For general $\lambda \in \mathbb{C}$, similar conclusions can be made.

Define the **root condition**:

$$\begin{cases} |r_k(0)| \leq 1 & k = 0, 1, \dots, p, \\ \text{and any root with } |r_k(0)| = 1 \text{ has multiplicity one.} \end{cases}$$

A method is **stable** if and only if the **root condition** is satisfied.

A method is said to be stable for a given problem if it is robust against perturbation for sufficiently small stepsize $h_0 > 0$. To determine this maximum stable stepsize h_0 , we define the region of absolute stability.

The **region of absolute stability** is the set of all $z = h\lambda$ such that $|r_k(z)| < 1$ for all roots.

For the model problem $y' = \lambda y$, we should then choose h small enough such that $h\lambda$ is inside this region.

Examples

Consider the trapezoidal method:

$$w_{i+1} = w_i + \frac{h}{2} [f(t_{i+1}, w_{i+1}) + f(t_i, w_i)].$$

In the form of (2), we have $p = 0$, $a_0 = 1$, $b_{-1} = \frac{1}{2}$, $b_1 = \frac{1}{2}$. The method is consistent since

$$\sum_{k=0}^p a_k = a_0 = 1, \quad -\sum_{k=0}^p k a_k + \sum_{k=-1}^p b_k = 0 + \frac{1}{2} + \frac{1}{2} = 1.$$

Its characteristic equation is

$$\begin{aligned} (1 - b_{-1}z)r - (a_0 + b_0z) &= 0 \\ (1 - \frac{1}{2}z)r - (1 + \frac{1}{2}z) &= 0, \end{aligned}$$

where $z = h\lambda$. When $z = 0$, it reduces to $r - 1 = 0$. Its only root is $r_0(0) = 1$, which satisfies the root condition. Thus the method is stable. Since it is both consistent and stable, it is convergent. The region of stability is where $|r_0(z)| < 1$:

$$\begin{aligned} |r_0(z)| &= \left| \frac{1 + \frac{1}{2}z}{1 - \frac{1}{2}z} \right| < 1 \\ |1 + \frac{1}{2}z| &< |1 - \frac{1}{2}z| \\ |z + 2| &< |z - 2| \\ |z + 2|^2 &< |z - 2|^2 \\ (x + 2)^2 + y^2 &< (x - 2)^2 + y^2 \quad z = x + iy \\ 4x &< -4x \\ \Rightarrow x &< 0 \end{aligned}$$

The region of stability is the left half-plane, $\text{Re } z < 0$. A method with the left half-plane in its stability region is called **A-stable**.

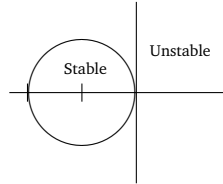
Forward Euler's method is

$$w_{i+1} = w_i + hf(t_i, w_i).$$

We can rewrite as in (2) with $p = 0$, $a_0 = 1$, $b_{-1} = 0$, $b_0 = 1$. Its characteristic equation is

$$r - (1 + z) = 0,$$

where $z = h\lambda$. We see that it is stable by noting that for $z = 0$, its only root is $r_0(0) = 1$, which satisfies the root condition. The region of stability is $|1 + z| < 1$, pictured below.



Backward (implicit) Euler's method is

$$w_{i+1} = w_i + hf(t_{i+1}, w_{i+1}).$$

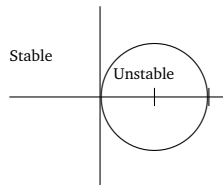
In the form of (2), we have $p = 0$, $a_0 = 1$, $b_{-1} = 1$, $b_0 = 0$. Its characteristic equation is

$$(1 - z)r - 1 = 0.$$

For $z = 0$ it is $r - 1 = 0$ with the root $r_0(0) = 1$. It satisfies the root condition, so the method is stable. The region of absolute stability is

$$\left| \frac{1}{1 - z} \right| < 1 \quad \iff \quad 1 < |1 - z|$$

Backward Euler's method is A-stable.



Consider the method

$$w_{i+1} = 4w_i - 3w_{i-1} - 2hf(t_{i-1}, w_{i-1}),$$

with $p = 1$, $a_0 = 4$, $a_1 = -3$, $b_{-1} = 0$, $b_0 = 0$, $b_1 = -2$. This method is consistent, since

$$\sum_{k=0}^p a_k = 4 - 3 = 1, \quad -\sum_{k=0}^p ka_k + \sum_{k=-1}^p b_k = 0 + b_1 = 1.$$

For $h\lambda = 0$, its characteristic equation is

$$\begin{aligned} r^2 - a_0r - a_1 &= 0 \\ r^2 - 4r + 3 &= 0, \end{aligned}$$

which has roots $r_0(0) = 1$ and $r_1(0) = 3$. Since $r_1(0)$ is outside of the unit circle, the root condition is not satisfied and the method is unstable. Hence it is also not convergent.

Thus we see that it is possible for a method to be consistent but not convergent.

We seek an explicit fourth-order method of the form

$$w_{i+1} = a_0w_i + a_1w_{i-1} + a_2w_{i-2} + h[b_0f(t_i, w_i) + b_1f(t_{i-1}, w_{i-1}) + b_2f(t_{i-2}, w_{i-2})].$$

To be fourth order requires $a_0 = \gamma$, $a_1 = 9$, $a_2 = -8 - \gamma$, $b_0 = \frac{1}{3}(9 - \gamma)$, $b_1 = -\frac{2}{3}(9 + 2\gamma)$, $b_2 = -\frac{1}{3}(9 + \gamma)$. The characteristic equation for $h\lambda = 0$ is

$$r^3 - \gamma r^2 - 9r + (8 + \gamma) = 0,$$

which has roots

$$r_0(0) = 1, \quad r_1(0) = \frac{1}{2}(\gamma - 1 - \sqrt{\gamma^2 + 2\gamma + 33}), \quad r_2(0) = \frac{1}{2}(\gamma - 1 + \sqrt{\gamma^2 + 2\gamma + 33}).$$

However, since $\gamma^2 + 2\gamma + 33 \geq 32$ for all $\gamma \in \mathbb{R}$, we have $|r_1(0) - r_2(0)| = \sqrt{\gamma^2 + 2\gamma + 33} \geq 4\sqrt{2} > 2$. For any γ , we cannot have both $r_1(0)$ and $r_2(0)$ simultaneously inside the unit circle, so there is no γ such that the method is stable.

Therefore, there are no three-step explicit and stable fourth-order methods.

The Adams-Moulton two-step (implicit) method is

$$w_{i+1} = w_i + \frac{h}{12} [5f(t_{i+1}, w_{i+1}) + 8f(t_i, w_i) - f(t_{i-1}, w_{i-1})].$$

In the form (2), we have $p = 1$, $a_0 = 1$, $a_1 = 0$, $b_{-1} = \frac{5}{12}$, $b_0 = \frac{8}{12}$, $b_1 = -\frac{1}{12}$. The method is consistent since

$$\sum_{k=0}^p a_k = a_0 = 1, \quad -\sum_{k=0}^p k a_k + \sum_{k=-1}^p b_k = 0 + \frac{5}{12} + \frac{8}{12} - \frac{1}{12} = 1.$$

To check that this method is stable, we see that the characteristic equation for $h\lambda = 0$ is

$$r^2 - a_0 r - a_1 = 0 \quad \implies \quad r^2 - r = 0,$$

which has roots $r_0(0) = 1$ and $r_1(0) = 0$. The root condition is satisfied, so the method is indeed stable. To find the region of absolute stability,

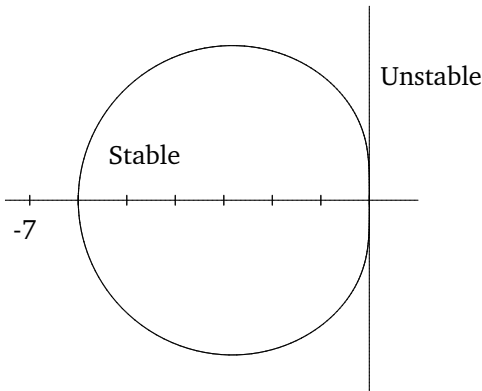
$$\begin{aligned} (1 - b_{-1}z)r^2 - (a_0 + b_0z)r - (a_1 + b_1z) &= 0 \\ (1 - \frac{5}{12}z)r^2 - (1 + \frac{8}{12}z)r + \frac{1}{12}z &= 0 \end{aligned}$$

In principle, we could compute both roots $r_0(z)$, $r_1(z)$ and then find for which z both are inside the unit circle, but that is hard.

Instead, solving for z , we have $z = \frac{12r(r-1)}{5r^2+8r-1}$. We can find the boundary of the region of absolute stability by substituting $r = e^{i\theta}$. As a function of θ ,

$$z(\theta) = \frac{12e^{i\theta}(e^{i\theta} - 1)}{5e^{i2\theta} + 8e^{i\theta} - 1}$$

parameterizes the boundary of the region of absolute stability. (You can use this trick for any method with the form (2).)



The Adams-Bashforth four-step (explicit) method is

$$w_{i+1} = w_i + \frac{h}{24} [55f(t_i, w_i) - 59f(t_{i-1}, w_{i-1}) + 37f(t_{i-2}, w_{i-2}) - 9f(t_{i-3}, w_{i-3})],$$

with $p = 3$ and $a_0 = 1, a_1 = 0, a_2 = 0, b_{-1} = 0, b_0 = \frac{55}{24}, b_1 = -\frac{59}{24}, b_2 = \frac{37}{24}, b_3 = -\frac{9}{24}$. Its characteristic equation is

$$\begin{aligned} r^4 - (a_0 + b_0z)r^3 - (a_1 + b_1z)r^2 - (a_2 + b_2z)r - (a_3 + b_3z) &= 0 \\ r^4 - (1 + \frac{55}{24}z)r^3 + \frac{59}{24}zr^2 - \frac{37}{24}zr + \frac{9}{24}z &= 0. \end{aligned}$$

When $z = 0$, we have $r^4 - r^3 = 0$, which has roots $r_0(0) = 1$ and $r_1(0) = r_2(0) = r_3(0) = 0$. So the method satisfies the root condition and is stable. Using the same trick as on the previous example, we can find the boundary of the region of stability from

$$z(\theta) = \frac{24(e^{i4\theta} - e^{i3\theta})}{55e^{i3\theta} - 59e^{i2\theta} + 37e^{i\theta} - 9}.$$

(In this case, the curve intersects itself twice, and two of the loops are unstable regions—these excess loops have been removed in the plot below.)

