F'06. *Let Ω be an open, bounded and connected subset of $\mathbb{R}^2$, with sufficiently smooth boundary. Consider the problem*

$$\begin{cases} -\dfrac{\partial}{\partial x}\left((1+2x^2+3y^4)u_x\right) - u_{yy} &= f \text{ in } \Omega \\ (1+2x^2+3y^4)u_x n_x + u_y n_y + \lambda u &= g \text{ on } \Gamma = \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$, $g \in L^2(\Gamma)$, $\vec{n} = (n_x, n_y)$ is the outward normal to $\partial\Omega$, and $\lambda \geq 0$ is a constant.

(a) Give weak variational formulations of the problem, by considering the cases $\lambda = 0$ and $\lambda > 0$. Show that each of these formulations has one and only solution (under additional conditions on u, f, or g if necessary, that you will specify).

(a) Let $\sigma = \begin{pmatrix} (1+2x^2+3y^4) & 0 \\ 0 & 1 \end{pmatrix}$ such that the problem may be rewritten as

$$\begin{cases} -\nabla \cdot (\sigma \cdot \nabla u) &= f \text{ in } \Omega \\ n \cdot \sigma \cdot \nabla u + \lambda u &= g \text{ on } \Gamma = \partial\Omega \end{cases}$$

Let $V = H^1(\Omega)$. For any $v \in V$,

$$\int_\Omega -\nabla \cdot (\sigma \cdot \nabla u) v = \int_\Omega fv$$
$$\int_\Omega \nabla v \cdot \sigma \cdot \nabla u - \int_\Gamma v(n \cdot \sigma \cdot \nabla u) = \int_\Omega fv$$
$$\int_\Omega \nabla v \cdot \sigma \cdot \nabla u + \int_\Gamma \lambda uv = \int_\Omega fv + \int_\Gamma gv.$$

Let $a(u,v) = \int_\Omega \nabla v \cdot \sigma \cdot \nabla u + \int_\Gamma \lambda uv$ and $L(v) = \int_\Omega fv + \int_\Gamma gv$, then the weak formulation is: find $u \in V$ such that $a(u,v) = L(v)$ for all $v \in V$.

Let C be such that $\|v\|_{L^2(\Gamma)} \leq C \|v\|_{H^1(\Omega)}$. If we assume $g \in L^2(\Gamma)$, $L(v)$ is continuous:

$$|L(v)| \leq \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} \|v\|_{L^2(\Gamma)} \leq \left(\|f\|_{L^2(\Omega)} + C \|g\|_{L^2(\Omega)}\right) \|v\|_{H^1(\Omega)} .$$

Let $B = \sup_{(x,y)\in\Omega} \left|1 + 2x^2 + 3y^4\right|$, then we have that a is continuous

$$|a(u,v)| \leq B \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \lambda \|u\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \leq (B + \lambda C^2) \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} .$$

F'05. *The following elliptic problem is approximated by the finite element method,*

$$\begin{cases} -\nabla\cdot(a(\vec{x})\nabla u(\vec{x})) &= f(\vec{x}), \quad \vec{x}\in\Omega=(0,1)\times(0,1),\\ u(\vec{x}) &= u_0(\vec{x}), \quad \vec{x}\in\Gamma_1=\{(x_1,x_2): x_1=0,\ 0\le x_2\le 1\},\\ \frac{\partial u(\vec{x})}{\partial x_1}+u(\vec{x}) &= 0, \quad \vec{x}\in\Gamma_2=\{(x_1,x_2): x_1=1,\ 0\le x_2\le 1\},\\ \frac{\partial u(\vec{x})}{\partial x_2} &= 0, \quad \vec{x}\in\Gamma_3=\{(x_1,x_2): 0<x_1<1,\ x_2=0,1\}.\end{cases}$$

where $0<A\le a(x)\le B$ *for a.e.* $x\in\Omega$, $f\in L^2(\Omega)$, *and* $u_{0|\Gamma_1}$ *is the trace of a function* $u_0\in H^1(\Omega)$.

(a) Determine an appropriate weak variational formulation of the problem.

(b) Prove conditions on the corresponding linear and bilinear forms which are needed for existence and uniqueness of the solution.

(c) Setup a finite element approximation using P_1 *elements, and a set of basis functions such that the associated linear system is sparese and of band structure. Discuss the linear system thus obtained, and give the rate of convergence.*

(a) Reformulate the problem as $u=\phi+u_0$, $\phi\in V=\{v\in H^1(\Omega): v_{|\Gamma_1}=0\}$,

$$\begin{cases} -\nabla\cdot(a\nabla\phi) &= \tilde{f} \quad \text{in }\Omega,\\ n\cdot\nabla\phi+\phi &= h \quad \text{in }\Gamma_2,\\ n\cdot\nabla\phi &= g \quad \text{in }\Gamma_3,\end{cases}$$

where $\tilde{f}=f+\nabla\cdot(a\nabla u_0)$, $h=-n\cdot\nabla u_0-u_0$, and $g=-n\cdot\nabla u_0$. Assuming that $a\in L^\infty(\Omega)$, we have $\tilde{f}\in H^{-1}(\Omega)$ and $u_0\in H^1(\Omega)$ implies $h\in H^{\frac{1}{2}}(\Gamma_2)$ and $g\in H^{\frac{1}{2}}(\Gamma_3)$. For $v\in V$,

$$\int_\Omega -\nabla\cdot(a\nabla\phi)v=\int_\Omega \tilde{f}v$$

$$\int_\Omega a\nabla v\cdot\nabla\phi-\int_{\Gamma_1}\underbrace{a(n\cdot\nabla\phi)v}_{=0}-\int_{\Gamma_2}\underbrace{a(n\cdot\nabla\phi)v}_{=a(h-\phi)v}-\int_{\Gamma_3}\underbrace{a(n\cdot\nabla\phi)v}_{=agv}=\int_\Omega\tilde{f}v$$

$$\int_\Omega a\nabla v\cdot\nabla\phi+\int_{\Gamma_2}a\phi v=\int_\Omega\tilde{f}v+\int_{\Gamma_2}ahv+\int_{\Gamma_3}agv.$$

Let $b(\phi,v)=\int_\Omega a\nabla v\cdot\nabla\phi+\int_{\Gamma_2}a\phi v$ and $L(v)=\int_\Omega\tilde{f}v+\int_{\Gamma_2}ahv+\int_{\Gamma_3}agv$, then the weak formulation is: find $u=\phi+u_g\in H^1(\Omega)$, $\phi\in V$, such that $b(\phi,v)=L(v)$ for all $v\in V$.

(b) Since $v_{|\Gamma_1}=0$ and Γ_1 has positive length, there exists $C_1>0$ such that $\|\nabla v\|_{L^2(\Omega)}\ge C_1\|v\|_{H^1(\Omega)}$. Therefore, b is coercive:

$$b(v,v)=\int_\Omega a|\nabla v|^2+\int_{\Gamma_2}a|v|^2\ge A\|\nabla v\|^2_{L^2(\Omega)}\ge AC_1^2\|v\|^2_{H^1(\Omega)}$$

By Cauchy-Schwarz, $|b(\phi,v)|\le B\|\nabla v\|_{L^2(\Omega)}\|\nabla\phi\|_{L^2(\Omega)}+B\|\phi\|_{L^2(\Gamma_2)}\|v\|_{L^2(\Gamma_2)}$. Since $\|v\|^2_{H^1(\Omega)}=\|v\|^2_{L^2(\Omega)}+\|\nabla v\|^2_{L^2(\Omega)}$, we have $\|\nabla v\|_{L^2(\Omega)}\le\|v\|_{H^1(\Omega)}$. There exists a constant $C_2>0$ such that $\|v\|_{L^2(\Gamma_2)}\le C_2\|v\|_{H^1(\Omega)}$. Therefore, b is continuous:

$$|b(\phi,v)|\le B\big(1+C_2\big)\|\phi\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)}.$$

Let $C_3>0$ be such that $\|v\|_{L^2(\Gamma_3)}\le C_3\|v\|_{H^1(\Omega)}$, then L is continuous since

$$\begin{aligned}|L(v)|&\le\|f\|_{H^{-1}(\Omega)}\|v\|_{H^1(\Omega)}+B\|h\|_{L^2(\Gamma_2)}\|v\|_{L^2(\Gamma_2)}+B\|g\|_{L^2(\Gamma_3)}\|v\|_{L^2(\Gamma_3)}\\&\le\big(\|f\|_{H^{-1}(\Omega)}+BC_2\|h\|_{L^2(\Gamma_2)}+BC_3\|g\|_{L^2(\Gamma_3)}\big)\|v\|_{H^1(\Omega)}.\end{aligned}$$

Thus by the Lax-Milgram lemma, the weak formulation has a unique solution.

W'05. *Consider the boundary value problem*

$$\begin{cases} -\Delta u + u = f(x,y), & \text{for } (x,y) \in \Omega = [0,1]\times[0,1] \\ u = 0, & \text{for } (x,y) \in \partial\Omega,\ x = 0,1 \\ u_y = 0, & \text{for } (x,y) \in \partial\Omega,\ y = 0,1 \end{cases}$$

(a) Give a weak variational formulation of the problem.

(b) Analyze the existence and uniqueness of the solution to this problem. Assume $f \in L^2(\Omega)$.

(c) Formulate a finite element approximation of the elliptic problem using piecewise-linear elements. Discuss the form and properties of the stiffness matrix and the existence and uniqueness of the solution of the linear system thus obtained.

(a) Let $\Gamma_1 = \partial\Omega \cap \{x = 0,1\}$ and $\Gamma_2 = \partial\Omega \cap \{y = 0,1\}$. Let $V = \{v \in H^1(\Omega) : v_{|\Gamma_1} = 0\}$. For any $v \in V$,

$$\int_\Omega -\Delta u v + uv = \int_\Omega fv$$

$$\int_\Omega \nabla v \cdot \nabla u + uv - \int_{\Gamma_1} \underbrace{v(n\cdot\nabla u)}_{\text{zero since } v_{|\Gamma_1} = 0} - \int_{\Gamma_2} \underbrace{v(n\cdot\nabla u)}_{\text{zero since } u_y = 0 \text{ on } \Gamma_2} = \int_\Omega fv$$

Let $a(u,v) = \int_\Omega \nabla v \cdot \nabla u + uv$ and $L(v) = \int_\Omega fv$. The weak formulation is: find $u \in V$ such that $a(u,v) = L(v)$ for all $v \in V$.

(b) Since $a(u,v) = \langle u, v\rangle_{H^1(\Omega),H^1(\Omega)}$, it is coercive and by Cauchy-Schwarz it is continuous. The linear form $L(v)$ is continuous, also by Cauchy-Schwarz:

$$|L(v)| \le \|f\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)} \le \|f\|_{L^2(\Omega)} \|v\|_{H^1(\Omega)}.$$

By the Lax-Milgram lemma, the problem has a unique solution.

F'04. *Let Ω be a sufficiently smooth and bounded domain in the plane and let the boundary Γ of Ω be divided into two parts Γ_1 and Γ_2. Give a variational formulation of the following problem:*

$$\begin{cases} -\Delta u + u &= f \quad \text{in } \Omega, \\ n \cdot \nabla u &= g \quad \text{on } \Gamma_1, \\ u &= u_0 \quad \text{on } \Gamma_2, \end{cases}$$

where f, u_0 and g are given functions satisfying some appropriate assumptions (that you should specify). Formulate a FEM for this problem, and discuss (verify) the assumptions of the Lax-Milgram lemma.

W'04. *Consider the following problem in a domain $\Omega \subset \mathbb{R}^2$ with $\Gamma = \partial\Omega$.*

$$\begin{cases} -\Delta u + \beta_1 \frac{\partial u}{\partial x_1} + \beta_2 \frac{\partial u}{\partial x_2} + u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{in } \Gamma \end{cases}$$

where β_1 and β_2 are constants.

(a) Choose an appropriate space of test functions V, and give a weak formulation of the problem.

(b) For any $v \in V$, show that

$$\int_\Omega \left(\beta_1 \frac{\partial v}{\partial x_1} v + \beta_2 \frac{\partial v}{\partial x_2} v\right) \,\mathrm{d}x = 0.$$

(c) By analyzing the linear and bilinear forms, show that the weak formulation has a unique solution.

(d) Set up a convergent finite element approximation and discuss the linear system thus obtained.

(a) Set $V = H_0^1(\Omega) = \{v \in L^2(\Omega) : |\nabla v| \in L^2(\Omega), v_{|\Gamma} = 0\}$ and let $\beta = \binom{\beta_1}{\beta_2}$. For any $v \in V$,

$$\int_\Omega -\Delta u\, v + (\beta \cdot \nabla u)v + uv = \int_\Omega fv$$

$$\int_\Omega \nabla v \cdot \nabla u + (\beta \cdot \nabla u)v + u\,v - \underbrace{\int_\Gamma (n \cdot \nabla u)v}_{=0} = \int_\Omega fv.$$

Let $a(u,v) = \int_\Omega \nabla v \cdot \nabla u + (\beta \cdot \nabla u)v + uv$ and $L(v) = \int_\Omega fv$. The weak formulation is: find $u \in V$ such that $a(u,v) = L(v)$ for all $v \in V$.

(b) For any $v \in V$,

$$\int_\Omega \left(\beta_1 \frac{\partial v}{\partial x_1} v + \beta_2 \frac{\partial v}{\partial x_2} v\right) \,\mathrm{d}x = \int_\Omega v\beta \cdot \nabla v$$

$$= -\int_\Omega v\nabla \cdot (v\beta) + \int_\Gamma v(v\beta) \cdot n = -\int_\Omega v\left(\frac{\partial v}{\partial x_1}\beta_1 + \frac{\partial v}{\partial x_2}\beta_2\right) = -\int_\Omega v\beta \cdot \nabla v.$$

Therefore, $\int_\Omega v\beta \cdot \nabla v = 0$.

(c) $V = H_0^1(\Omega)$ is a Hilbert space with inner product $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v + uv$. If $f \in H^{-1}(\Omega) = (H^1(\Omega))'$, then $L(v) = \langle f, v\rangle_{H^{-1}, H_0^1}$ is a continuous linear form. By part (b), we have

$$a(v,v) = \int_\Omega \nabla v \cdot \nabla v + vv = \|v\|_{H_0^1}^2,$$

so a is coercive. Let $\|\beta\|_{\ell^\infty}$ denote $\max\{|\beta_1|, |\beta_2|\}$. By Cauchy-Schwarz,

$$|a(u,v)| = \left|\langle u, v\rangle_{H_0^1, H_0^1} + \int_\Omega (\beta \cdot \nabla u)v\right| \le \|u\|_{H_0^1} \|v\|_{H_0^1} + \|\beta\|_{\ell^\infty} \|v\|_{L^2} \|\nabla u\|_{L^2}.$$

Since $\|v\|_{H_0^1}^2 = \|v\|_{L^2}^2 + \|\nabla v\|_{L^2}^2$, we have $\|v\|_{L^2} \le \|v\|_{H_0^1}$ and $\|\nabla v\|_{L^2} \le \|v\|_{H_0^1}$, which yields

$$|a(u,v)| \le (1 + \|\beta\|_{\ell^\infty}) \|u\|_{H_0^1} \|v\|_{H_0^1}.$$

So a is a continuous bilinear form. Therefore, by the Lax-Milgram lemma, the weak formulation has a unique solution.

F'03. *Develop and describe the piecewise-linear Galerkin finite element approximation of*

$$\begin{cases} -\Delta u + u &= f(x,y), \quad (x,y) \in T \ = \{(x,y) : x > 0,\ y > 0,\ x + y < 1\}, \\ u &= g_1(x), \quad (x,y) \in T_1 = \{(x,y) : y = 0,\ 0 < x < 1\}, \\ u &= g_2(y), \quad (x,y) \in T_2 = \{(x,y) : x = 0,\ 0 < y < 1\}, \\ \frac{\partial u}{\partial n} &= h(x,y), \quad (x,y) \in T_3 = \{(x,y) : x > 0,\ y > 0,\ x + y = 1\}. \end{cases}$$

Justify your approximation by analyzing the appropriate bilinear and linear forms. Give a weak formulation of the problem. Give a convergence estimate and quote the appropriate theorems for convergence.

Define $g \in$ on ∂T such that $g_{|T_1} = g_1$ and $g_{|T_2} = g_2$. Assume that g is smooth enough such that an extension $\tilde{g} \in H^{\frac{1}{2}}(\partial T)$ exists with $\tilde{g}_{|T_1 \cup T_2} = g$. Then there is a lifting $u_g \in H^1(T)$ such that $u_{g|\partial T} = \tilde{g}$. Reformulating the problem as $u = u_g + \phi$, $\phi \in H^1_{\partial T_D}(T) = \{v \in H^1(T) : v_{|T_1 \cup T_2} = 0\}$, the Dirichlet boundary conditions are homogeneous:

$$\begin{cases} -\Delta \phi + \phi &= f + \Delta u_g - u_g, \quad \text{in } T, \\ \phi &= 0, \quad \text{on } T_1 \cup T_2, \\ n \cdot \nabla \phi &= h - n \cdot \nabla u_g, \quad \text{on } T_3. \end{cases}$$

Let $\tilde{f} = f + \Delta u_g - u_g$ and $\tilde{h} = h - n \cdot \nabla u_g$. For any $v \in V = H^1_{\partial T_D}(T)$,

$$\int_T -\Delta \phi\, v + \phi\, v = \int_T \tilde{f} v$$

$$\int_T \nabla v \cdot \nabla \phi + \phi v - \int_{T_1 \cup T_2} \underbrace{(n \cdot \nabla \phi) v}_{=0} - \int_{T_3} \underbrace{(n \cdot \nabla \phi) v}_{=\tilde{h} v} = \int_T \tilde{f} v.$$

Let $a(\phi, v) = \int_T \nabla v \cdot \nabla \phi + \phi v$ and $L(v) = \int_T \tilde{f} v + \int_{T_3} \tilde{h} v$, then the weak formulation is

$$\begin{cases} \text{Find } u \in H^1(T) \text{ such that} \\ u = u_g + \phi, \quad \phi \in V, \\ a(\phi, v) = L(v), \quad \forall v \in V. \end{cases}$$

V is a Hilbert space. Since $a(\phi, v) = \langle \phi, v \rangle_{V,V}$, is coercive and by Cauchy-Schwarz continuous. If $f, h \in H^{-1}(T)$, then $\tilde{f}, \tilde{h} \in H^{-1}(T)$ and L is continuous. Then by the Lax-Milgram lemma, the weak formulation has a unique solution.