Nonlinear First-Order Equations. Consider $F(x,y,u,p,q)=0$, where $p=u_x$ and $q=u_y$. The characteristic equations are

$$\begin{aligned} x' &= F_p \\ y' &= F_q \\ u' &= pF_p+qF_q \\ p' &= -F_x-pF_u \\ q' &= -F_y-qF_u \end{aligned} \qquad \text{and} \qquad \begin{aligned} F(x_0,y_0,u_0,p_0,q_0) &= 0 \\ p_0x_0'+q_0y_0' &= u_0'. \end{aligned}$$

Solution of the Wave Equation. The solution to the wave equation

$$\begin{cases} u_{tt}(x,t)-c^2\Delta u(x,t) = f(x,t) \\ u(x,0) = g(x) \\ u_t(x,0) = h(x) \end{cases}$$

with $f=0$, $g\in C^2(\mathbb{R})$, $h\in C^1(\mathbb{R})$ in 1D is d'Alembert's solution,

$$u(x,t)=\tfrac{1}{2}(g(x-ct)+g(x+ct))+\tfrac{1}{2c}\int_{x-ct}^{x+ct}h(\xi)\,\mathrm{d}\xi.$$

If $f\neq 0$, $f\in C^{1;0}$, use Duhamel's principle: for each s, solve

$$\begin{cases} U_{tt}-c^2U_{xx} = 0 \\ U(x,0,s) = 0 \\ U_t(x,0,s) = f(x,s) \end{cases}$$

and then add $\displaystyle\int_0^t U(x,t-s,s)\,\mathrm{d}s$ to d'Alembert's solution.

Wave equation in 3D: the solution with $f=0$, $g\in C^3(\mathbb{R}^3)$, $h\in C^2(\mathbb{R}^3)$ is

$$u(x,t)=\frac{1}{4\pi}\frac{\partial}{\partial t}\Big(t\int_{|\xi|=1}g(x+ct\xi)\,\mathrm{d}S_\xi\Big)+\frac{t}{4\pi}\int_{|\xi|=1}h(x+ct\xi)\,\mathrm{d}S_\xi.$$

Wave equation in 2D: the solution with $f=0$, $g\in C^3(\mathbb{R}^2)$, $h\in C^2(\mathbb{R}^2)$ is

$$u(x,t)=2\left[\frac{1}{4\pi}\frac{\partial}{\partial t}\Big(t\int_{|\xi|<1}\frac{g(x+ct\xi)\,\mathrm{d}\xi}{\sqrt{1-|\xi|^2}}\Big)+\frac{t}{4\pi}\int_{|\xi|<1}\frac{h(x+ct\xi)\,\mathrm{d}\xi}{\sqrt{1-|\xi|^2}}\right].$$

Solution of the Laplace Equation. The fundamental solution to $\Delta K(x) = \delta(x)$ is

$$K(x) = \begin{cases} \dfrac{1}{2\pi} \log |x| & \text{if } n = 2 \\ \dfrac{1}{(2-n)\omega_n} |x|^{2-n} & \text{if } n \geq 3 \end{cases}$$

Let $G(x,\xi) = K(x-\xi) + \omega_\xi(x)$, where $\omega_\xi(x)$ is any harmonic function in Ω, then

$$u(\xi) = \int_\Omega G(x,\xi)\Delta u \, \mathrm{d}x + \int_{\partial\Omega} \left(u(x) \frac{\partial G(x,\xi)}{\partial \nu_x} - G(x,\xi) \frac{\partial u(x)}{\partial \nu} \right) \mathrm{d}S_x.$$

If $\omega_\xi(x) = -K(x-\xi)$ for all $x \in \partial\Omega$, G is the Green's function.

If Ω is a bounded domain satisfying the exterior cone condition, then the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$.

Dirichlet problem on a half-space: if $\Omega = \mathbb{R}^n \cap \{x_n > 0\}$, the solution is $\partial\Omega$ has solution

$$u(\xi) = \frac{2\xi_n}{\omega_n} \int_{\mathbb{R}^{n-1}} \frac{g(x')}{|x'-\xi|^n} \, \mathrm{d}x' \qquad (x' \in \mathbb{R}^{n-1} \text{ is identified with } (x',0) \in \mathbb{R}^n).$$

Dirichlet problem on a ball: if $\Omega = B_a(0)$, the solution is

$$u(\xi) = \frac{a^2 - |\xi|^2}{a\omega_n} \int_{|x|=a} \frac{g(x)}{|x-\xi|^n} \, \mathrm{d}S_x.$$

Solution of the Heat Equation. Let Ω be a bdd domain and let (λ_n, ϕ_n) denote the eigenvalues and normalized eigenfunctions solving

$$\begin{cases} \Delta \phi_n = \lambda_n \phi_n & \text{in } \Omega \\ \phi_n = 0 & \text{on } \partial\Omega \end{cases}$$

Define the heat kernel $K(x,y,t) = \sum_{n=1}^\infty \mathrm{e}^{-\lambda_n t} \phi_n(x)\phi_n(y)$, then the heat equation

$$\begin{cases} u_t = \Delta u & x \in \Omega,\ t > 0 \\ u(x,0) = g(x) & x \in \overline{\Omega} \\ u(x,t) = 0 & x \in \partial\Omega,\ t > 0 \end{cases}$$

has solution

$$u(x,t) = \int_\Omega K(x,y,t) g(y) \, \mathrm{d}y.$$

Define $U = \Omega \times (0,T)$. If $u, v \in C^{2;1}(U) \cap C(\overline{U})$ are both solutions, then $u \equiv v$.

If $\Omega = \mathbb{R}^n$ and $g \in C(\mathbb{R}^n)$ is bdd, $K(x,y,t) = \dfrac{1}{(4\pi t)^{n/2}} \exp\left(-\frac{|x-y|^2}{4t}\right)$.

Harmonic Properties. Suppose $u \in C^2(\Omega)$ is harmonic, then the following hold:
(i) $u \in C^\infty(\Omega)$
(ii) The mean-value property holds:

$$u(\xi) = M_u(\xi, r) = \frac{1}{\omega_n} \int_{|x|=1} u(\xi + rx)\, \mathrm{d}S_x = \frac{n}{\omega_n} \int_{|x|\leq 1} u(\xi + rx)\, \mathrm{d}S_x \quad \text{if } \overline{B_r(\xi)} \subset \Omega.$$

(iii) (Harnack Inequality) Suppose also $u \geq 0$. If Ω_1 is a bdd domain, $\overline{\Omega_1} \subset \Omega$, then $\exists C_1$ independent of u such that

$$\sup_{x\in\Omega_1} u(x) \leq C_1 \inf_{x\in\Omega_1} u(x)$$

Subharmonic Properties. If $u \in C(\Omega)$ is subharmonic, $u(\xi) \leq M_u(\xi, r)$, the following hold:
(i) Either u is a constant or $u(x) < \sup_{\xi\in\Omega} u(\xi)$ for all $x \in \Omega$.
(ii) If $u \in C^2(\Omega)$, then $\Delta u \geq 0$ in Ω.

Uniform Ellipticity. An operator $L : H^1(\Omega) \to \mathbb{R}$,

$$Lu = A(x)\nabla^2 u + b(x) \cdot \nabla u + c(x),$$

with bounded coefficients A, b, c is uniformly elliptic if $A(x)$ is nonnegative definite for all $x \in \Omega$.

Weak Elliptic Maximum Principle. Suppose $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \geq 0$, where L is uniformly elliptic with $c(x) = 0$. Then $\max_{x\in\overline{\Omega}} u(x) \leq \max_{x\in\partial\Omega} u(x)$.

Strong Elliptic Maximum Principle. Suppose $u \in C^2(\Omega)$. Let $M = \sup_{x\in\Omega} u(x) < \infty$.
(i) If $u(x_0) = M$ for $x_0 \in \Omega$, then u is constant.
(ii) If u is not constant and $u(x_0) = M$ for $x_0 \in \partial\Omega$, then if $\frac{\partial u}{\partial \nu}(x_0)$ exists, $\frac{\partial u}{\partial \nu}(x_0) > 0$.