4. *Consider the Laplace equation*

$$u_{xx} + u_{yy} = 0, \quad y > 0,\ x \in \mathbb{R}$$

together with the boundary condition

$$u_y(x,0) - u(x,0) = f(x),$$

where $f(x) \in C_0^\infty$. *Find a representation for a bounded solution and show* $u(x,y) \to 0$ *as* $y \to \infty$, *uniformly in* x.

Take Fourier transforms over x to obtain

$$\begin{cases} -\omega^2 \hat{u}(\omega,y) + \hat{u}_{yy}(\omega,y) &= 0 \\ \hat{u}_y(\omega,0) - \hat{u}(\omega,0) &= \hat{f}(\omega). \end{cases}$$

For each fixed ω, this is an ODE with (bounded) solution

$$\hat{u}(\omega,y) = c(\omega)\mathrm{e}^{-|\omega|y}.$$

The boundary condition implies

$$c(\omega) = -\frac{\hat{f}(\omega)}{1+|\omega|},$$

so $\hat{u}(\omega,y) = -\mathrm{e}^{-|\omega|y}\hat{f}(\omega)/(1+|\omega|)$. Since f is in the Schwartz space, $\hat{f}$ is in the Schwartz space, and thus $\hat{u}(\omega,y)$ and $u(x,y)$ are in the Schwartz space for each fixed y. Together with the exponential decay in y, this implies that $u(x,y)$ is bounded.

To show that $u(x,y) \to 0$ uniformly as $y \to \infty$, compute

$$\begin{aligned} \lim_{y\to\infty} \|u(\cdot,y)\|_\infty &\le \lim_{y\to\infty} \frac{1}{2\pi} \int_{\mathbb{R}} \left| \frac{-\mathrm{e}^{-|\omega|y}}{1+|\omega|} \hat{f}(\omega)\mathrm{e}^{i\omega x} \right| \mathrm{d}\omega \\ &= \lim_{y\to\infty} \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\mathrm{e}^{-|\omega|y}}{1+|\omega|} \left|\hat{f}(\omega)\right| \mathrm{d}\omega, \end{aligned}$$

and since the integrand is dominated by $\left|\hat{f}(\omega)\right| \in L^1$,

$$= \frac{1}{2\pi} \int_{\mathbb{R}} \lim_{y\to\infty} \frac{\mathrm{e}^{-|\omega|y}}{1+|\omega|} \left|\hat{f}(\omega)\right| \mathrm{d}\omega = 0.$$