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Math 164
Homework 5

6.1.3. The dual problem is:
minimize $w = 11y_1 + 23y_2 + 12y_3$
subject to

$$\begin{array}{rcl} 4y_1 + 3y_2 + 7y_3 & \geq & 6 \\ 3y_1 + 2y_2 + 4y_3 & \geq & -3 \\ -8y_1 + 7y_2 + 3y_3 & \leq & -2 \\ 7y_1 + 6y_2 + 2y_3 & = & 5 \\ y_1 \text{ free}, y_2 \leq 0, y_3 \geq 0 & & \end{array}.$$

It is easily verified that the dual of the dual problem is indeed the primal problem.

6.2.11. Suppose that the system $A^T y \leq 0, b^T y > 0$ has a solution, y^*. Assume, for the sake of contradiction, that the system $Ax = b, x \geq 0$ has a solution, x^*. Then, $b^T y^* = (x^{*T} A^T) y^* = x^{*T}(A^T y^*) \leq 0$, which is a contradiction. Hence, there is no solution for the said system.
Conversely, consider the problem minimize $0^T x$ subject to $Ax = b, x \geq 0$, which is infeasible. Hence, the dual problem, maximize $b^T y$ subject to $A^T y \leq 0, y$ free is unbounded. Now, we see that the feasible set, S, is nonempty since $0 \in S$. Since the problem is unbounded, for any M, there exists $y \in S$ such that $b^T y > M \geq 0$ and we are done.

6.2.12. The dual problem is:
maximize $w = y_1 + y_2$
subject to

$$\begin{array}{rcl} -2y_1 + y_2 & \leq & 2 \\ 2y_1 + 4y_2 & \leq & 9 \\ y_1 - y_2 & \leq & 3 \\ y_1, y_2 \geq 0 & & \end{array}$$

By inspection, the optimal solution is $(\frac{7}{2}, \frac{1}{2})$. Using complementary slackness, we see that the solution is $(0, \frac{1}{3}, \frac{1}{3}, 0, 0)$. Both solution yield optimal solution of 4.

10.2.3. We see that

$$\nabla f = \begin{pmatrix} 16x_1 + 3x_2 - 25 \\ 3x_1 + 14x_2 + 31 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

which yields that the only stationary point is $(\frac{443}{215}, -\frac{571}{215})$. Now, the Hessian is

$$\nabla^2 f = \begin{pmatrix} 16 & 3 \\ 3 & 14 \end{pmatrix},$$

which, by inspection, is positive definite, and so, we conclude that our stationary point is a local minimizer. We also see that f is actually bounded below, and so, the given point is a global minimizer. There are no local/global maximizers.