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Math 164
Homework 1

2.2.3. The set of local minimizers, obtained by inspecting the graph of the feasible set, is $\{(x_1, x_2) \mid x_1 = 1, x_2^2 \leq 3\} \cup \{(-2, 0)\}$. $(-2, 0)$ is the global minimizer.

2.2.8. Note that for $\hat{S}$ to have no interior points, every point has to be a boundary point. The points in question, the interior points of S, satisfy the constraint $x_1 + 2x_2 + 3x_3 = 6$. To make these points boundary points, we can "redefine" the plane as an intersection of regions over the plane and under the plane. In other words, consider the constraints $x_1 + 2x_2 + 3x_3 \geq 6$ and $x_1 + 2x_2 + 3x_3 \leq 6$. Defining $\hat{S}$ by these constraints along with $x_1, x_2, x_3 \geq 0$ gives us that every point in $\hat{S}$ is a boundary point.

2.3.2. Let $x = (x_1, x_2), y = (y_1, y_2) \in S_1$. Then, $x_1 + x_2 \leq 1$, $y_1 + y_2 \leq 1$, and $x_1, y_1 \geq 0$. Consider $\alpha x + (1 - \alpha)y = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2)$, where $0 < \alpha < 1$. We have $(\alpha x_1 + (1-\alpha)y_1) + (\alpha x_2 + (1-\alpha)y_2) = \alpha(x_1 + x_2) + (1-\alpha)(y_1 + y_2) \leq \alpha + (1 - \alpha) = 1$ and $\alpha x_1 + (1 - \alpha)y_1 \geq 0$, since $\alpha, 1 - \alpha, x_1, y_1$ are all nonnegative. Hence, $\alpha x + (1 - \alpha)y \in S_1$, and so, S_1 is convex.
Let $x = (x_1, x_2), y = (y_1, y_2) \in S_2$. Then, $x_1 - x_2 \geq 0$, $y_1 - y_2 \geq 0$, and $x_1, y_1 \leq 1$. Consider $\alpha x + (1 - \alpha)y = (\alpha x_1 + (1-\alpha)y_1, \alpha x_2 + (1-\alpha)y_2)$, where $0 < \alpha < 1$. Then, we have $(\alpha x_1 + (1-\alpha)y_1) - (\alpha x_2 + (1-\alpha)y_2) = \alpha(x_1 - x_2) + (1-\alpha)(y_1 - y_2) \geq 0$, since $\alpha, 1-\alpha, x_1 - x_2, y_1 - y_2$ are all nonnegative. We also have that $\alpha x_1 + (1-\alpha)y_1 \leq \alpha + (1-\alpha) = 1$. Hence, $\alpha x + (1-\alpha)y \in S_2$, and so, S_2 is convex.
Now, consider $(2, -1), (1, \frac{1}{2}) \in S$. Let us look at the convex combination $(\frac{3}{2}, -\frac{1}{4})$. Since $\frac{3}{2} + (-\frac{1}{4}) > 1$ and $\frac{3}{2} > 1$, we see that it is in neither S_1 nor S_2, and so, not in S. Thus, we see that S is not convex.

2.3.10. Let $x, y \in S$ and consider $\alpha x + (1 - \alpha)y$, where $0 < \alpha < 1$. Then, $g_i(x) > 0$ and $g_i(y) > 0$ for all $i \in [m]$. We see that, for any $i \in [m]$,

$$g_i(\alpha x + (1 - \alpha)y) \geq \alpha g_i(x) + (1 - \alpha)g_i(y) > 0,$$

since $\alpha g_i(x) > 0$ and $(1 - \alpha)g_i(y) > 0$. Thus, $\alpha x + (1 - \alpha)y \in S$, and so, S is convex.