1. Let $X$ be a proper variety over an algebraically closed field $k$. Prove that $\mathcal{O}_X(X) = k$.

*Hint:* Identify regular functions with morphisms to the affine line. Which assumptions on $X$ did you actually use in your proof?

2. Let $X$ be a scheme, $G$ a finite group which acts faithfully on $X$ (in other words, $G \subset \text{Aut}(X)$ is a finite subgroup). A *quotient of $X$ by $G$* is a scheme denoted by $X/G$ together with a morphism $\pi : X \to X/G$ such that for every scheme $Y$, the induced map

$$\text{hom}(X/G, Y) \cong \text{hom}(X, Y)^G$$

is a bijection. (Of course, the right hand side denotes the set of $f : X \to Y$ such that $f \circ \sigma = f$ for all $\sigma \in G$.)

(a) Let $A$ be a ring on which the finite group $G$ acts faithfully, and denote by $A^G \subset A$ the subring of $G$-invariants, $\pi : X = \text{Spec}(A) \to \text{Spec}(A^G)$ the induced morphism of affine schemes. Establish the following:

i. $G$ acts faithfully on $X$ by $\sigma \mapsto \text{Spec}(\sigma^{-1})$.

ii. Given two points $x, y \in X$, show that they lie in the same $G$-orbit if and only if $\pi(x) = \pi(y)$.

iii. $A$ is an integral extension of $A^G$.

iv. $\pi$ is a quotient map (on the underlying topological spaces).

v. $\pi : X = \text{Spec}(A) \to \text{Spec}(A^G)$ is a quotient of $X$ by $G$.

(b) (corrected:) Back to the general situation, prove that $X/G$ exists if every $x \in X$ admits an affine open neighborhood stable under $G$.

(c) Assume that $X/G$ exists and that $X$ is separated. Show that $X/G$ is separated.