1 Consider a morphism of schemes \( f : X \to Y \). Prove that the following conditions are equivalent:

(a) \( f \) is locally of finite type.

(b) For every affine open \( U \subset Y \), \( f^{-1}(U) \) has an open covering by affine opens \( (V_i) \) such that \( \mathcal{O}_X(V_i) \) is a finitely generated \( \mathcal{O}_Y(U) \)-algebra for all \( i \).

(c) For every affine open \( U \subset Y \) and every affine open \( V \subset f^{-1}(U) \), \( \mathcal{O}_X(V) \) is a finitely generated \( \mathcal{O}_Y(U) \)-algebra.

2 Let \( f : X \to Y \) be an immersion. Show that \( f \) can be factored as \( f = j \circ i \), where \( j \) is an open immersion, and \( i \) is a closed immersion.

Remark. It is not true in general that \( f \) also factors as \( f = i \circ j \) where \( i \) is a closed immersion, and \( j \) an open immersion. (Although this is true under mild assumptions.) Here is an example which you might want to think about (no need to hand it in).

Let \( k \) be a field and \( X = \mathbb{A}^\infty_k = \text{Spec}(k[x_1, x_2, \ldots]) \). Let \( j : U = \bigcup_{n \in \mathbb{N}} D(x_n) \to X \) be the open immersion. On \( D(x_n) \) consider the closed immersion \( Z_n \to D(x_n) \) defined by the ideal \( \langle x_n^1, x_n^2, \ldots, x_n^n - 1, x_{n+1}, x_{n+2}, \ldots \rangle \subset k[x_1, x_2, \ldots][1/x_n] \).

It is easy to see that these glue to a closed immersion \( i : Z \to U \). Hence we get an immersion \( f = j \circ i \). Why is there no factorization of \( f \) as an open immersion followed by a closed immersion? (Hint: use the description of closed subschemes of an affine scheme given in class.)

3 Let \( R \) be a ring and \( p \in \mathbb{N} \) a prime number. \( R \) is said to be of characteristic \( p \) if \( p \cdot 1_R = 0 \) in \( R \). A scheme \( X \) is said to be of characteristic \( p \) if for every open \( U \subset X \), \( \mathcal{O}_X(U) \) is of characteristic \( p \).

(a) Prove that the following conditions are equivalent:

i. \( X \) is of characteristic \( p \).

ii. For every \( x \in X \), \( \mathcal{O}_{X,x} \) is of characteristic \( p \).

iii. \( \mathcal{O}_X(X) \) is of characteristic \( p \).

iv. There exists a morphism of schemes \( X \to \text{Spec}(\mathbb{F}_p) \), where the latter denotes the finite field with \( p \) elements.

(b) Let \( X \) be a scheme of characteristic \( p \). Prove that there is a unique morphism of schemes \( F_p : X \to X \) such that

- \( F_p \) is the identity on the underlying topological space.
- For any open \( U \subset X \), \( F_p \) acts on \( \mathcal{O}_X(U) \) by \( r \mapsto r^p \).

\( F_p \) is called the absolute Frobenius.

(c) Let \( \overline{\mathbb{F}}_p \) be an algebraic closure of \( \mathbb{F}_p \). For a finite type \( \mathbb{F}_p \)-scheme \( X \), define a canonical morphism

\[ X(\mathbb{F}_p^e) \to X(\overline{\mathbb{F}}_p)^{F_p^e} \]

where the right hand side denotes the subset of morphisms \( f : \text{Spec}(\overline{\mathbb{F}}_p) \to X \) such that \( f = F_p \circ \cdots \circ F_p \circ f \) (\( F_p \) is composed \( e \) times). Prove that the map is bijective.