1 In class we defined the tangent space at \( p \), \( T_pX \), of a variety \( X \). It is natural to want to assemble all these spaces (for varying \( p \in X \)) together into the tangent bundle \( TX \) of \( X \). More precisely, \( TX \) should come with a morphism \( \pi : TX \to X \) whose fibers are the tangent spaces: \( \pi^{-1}(p) = T_pX \). We want to explore to what extent this construction can be performed inside the category of varieties.

(a) As a warm-up, define \( T\mathbb{A}^n \). Notice that it is indeed a variety.

(b) How would you define \( TX \) when \( X \) is an affine variety? Is \( TX \) a variety? What is a necessary condition on \( X \) for this possibly to be true?

(c) How would you approach the case of arbitrary varieties? What are the problems in carrying out your idea?

2 Recall that a function \( f : U \to \mathbb{C} \) on an open subset \( U \subset \mathbb{C} \) is analytic if for every \( z_0 \in U \) there exists \( \varepsilon > 0 \) and \( f_n \in \mathbb{C} \) such that

\[
f(z) = \sum_{n \geq 0} f_n(z - z_0)^n, \quad |z - z_0| < \varepsilon,
\]

where \( \sum_{n \geq 0} f_n z^n \) converges absolutely. \( f \) is invertible (i.e. \( 1/f \) is analytic) if and only if \( f(z) \neq 0 \) for all \( z \in U \).

(a) Define, for every open subset \( U \subset \mathbb{C} \), \( \mathcal{C}^\omega(U) \) to be the set of analytic functions on \( U \).

Show that, with the obvious restriction morphisms, this yields a sheaf of \( \mathbb{C} \)-algebras.

(b) Prove that the stalks of \( \mathcal{C}^\omega \) are local rings.

(c) For \( U \subset \mathbb{C} \) open define \( \mathcal{C}^{\omega,\times}(U) \subset \mathcal{C}^\omega(U) \) to be the set of invertible analytic functions.

Show that \( \mathcal{C}^{\omega,\times} \) is a sheaf of abelian groups and that the exponential function defines a morphism of sheaves of abelian groups \( \exp : \mathcal{C}^\omega \to \mathcal{C}^{\omega,\times} \).

(d) Show that the induced morphism \( \exp_{z_0} \) on stalks is surjective for all \( z_0 \in \mathbb{C} \).

(e) Show that \( \exp_U : \mathcal{C}^\omega(U) \to \mathcal{C}^{\omega,\times}(U) \) is not surjective in general.

3 Let \( X \) be a topological space, and let \( \mathcal{B} \) be a base for the topology of \( X \) which is closed under intersection. Every presheaf \( F \) on \( X \) restricts to a “presheaf \( F|_\mathcal{B} \) on \( \mathcal{B} \)” (i.e. a presheaf on the full subcategory of \( \text{Top}(X) \) spanned by \( \mathcal{B} \)).

(a) Formulate when a presheaf on \( \mathcal{B} \) should be called a sheaf.

(b) Prove that \( (\bullet)|_\mathcal{B} : \text{Sh}(X) \to \text{Sh}(\mathcal{B}) \) is an equivalence of categories. (If it isn’t then your definition in part (a) is not the right one...)

---

### Homework 3
Due: February 3, 2017

---