Homework 2
Solution to problem 4

4  (a) Let $A$ and $B$ be two affine $k$-algebras. Show that $A \otimes_k B$ is a coproduct of $A$ and $B$ in the category of affine $k$-algebras.

Proof. Denote by $i_A$ (resp. $i_B$) the canonical $k$-algebra morphism $A \to A \otimes_k B$ (resp. $B \to A \otimes_k B$). The image of $i_A$ and $i_B$ together generate $A \otimes_k B$ as a $k$-algebra hence $A \otimes_k B$ is finitely generated. I will prove below that $A \otimes_k B$ is a domain hence $A \otimes_k B \in \text{AffAlg}/k$.

Given an affine $k$-algebra $C$ and morphisms of $k$-algebras $\alpha : A \to C$ and $\beta : B \to C$, define $\gamma : A \times B \to C$ by $\gamma(a,b) = \alpha(a)\beta(b)$. Since $\alpha$ and $\beta$ are $k$-linear morphisms, $\gamma$ is $k$-bilinear and therefore passes to a $k$-linear morphism $\gamma : A \otimes_k B \to C$. Since $\alpha$ and $\beta$ are algebra morphisms, $\gamma$ is an algebra morphism as well.

By construction, we have $\gamma \circ i_A = \alpha$ and $\gamma \circ i_B = \beta$, and $\gamma$ is uniquely determined by these two conditions since the image of $i_A$ and $i_B$ generate $A \otimes_k B$ as a $k$-algebra. This says precisely that $A \otimes_k B$ is the coproduct of $A$ and $B$. □

Claim. Let $A$ and $B$ be affine $k$-algebras. Then $A \otimes_k B$ is an affine $k$-algebra as well.

Proof. We already remarked that $C = A \otimes_k B$ is finitely generated. Now let $c = \Sigma a_i \otimes b_i$ and $c' = \Sigma a_i' \otimes b_i'$ be elements of $C$ and assume $cc' = 0$. Rearranging terms we may assume that the $b_i$ (resp. the $b_i'$) are $k$-linearly independent. Denote by $I$ the ideal in $A$ generated by the $a_i$, and $I'$ the ideal in $A$ generated by the $a_i'$. Let $m \subset A$ be a maximal ideal. Under the morphism $(-) : A \otimes_k B \to A/m \otimes_k B \cong B$ (by the Nullstellensatz)\(^1\) we have $\bar{c} \bar{c'} = 0$ in $B$ hence $\bar{c} = 0$ or $\bar{c'} = 0$. So either $I \subset m$, or $I' \subset m$. Since $A$ is affine we must therefore have $I \subset \cap_{m \subset A} \text{maximal} m = 0$ or $I' \subset \cap_{m \subset A} \text{maximal} m = 0$ and so $c = 0$ or $c' = 0$. □

(b) Deduce that if $X \subset \mathbb{A}^m$ and $Y \subset \mathbb{A}^n$ are affine varieties then $X \times Y \subset \mathbb{A}^{m+n}$ with the induced topology is a product of $X$ and $Y$ in the category of varieties.

Proof. Let $\mathcal{O}(X) = k[x_1, \ldots, x_m]/I$ and $\mathcal{O}(Y) = k[y_1, \ldots, y_n]/J$ be the corresponding coordinate rings. By part (a), $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$ is a coproduct of $\mathcal{O}(X)$ and $\mathcal{O}(Y)$ in $\text{AffAlg}/k$ and hence a product in $(\text{AffAlg}/k)^\text{op}$. Under the adjunction

$$
\mathcal{O} : \text{Var}/k \rightleftarrows (\text{AffAlg}/k)^\text{op} : \text{mSpec}
$$

described in class, the right adjoint $\text{mSpec}$ takes products to products hence we reduce to prove that the coordinate ring of $X \times Y$ is isomorphic to $\mathcal{O}(X) \otimes_k \mathcal{O}(Y)$.

A polynomial $f \in R = k[x_1, \ldots, x_m, y_1, \ldots, y_n]$ vanishes on $X \times Y$ if and only if it vanishes on the intersection of $X \times \mathbb{A}^n$ and $\mathbb{A}^m \times Y$, i.e. if and only if $f \in \sqrt{R \cdot I + R \cdot J}$. Now, the canonical morphism

$$
k[x_1, \ldots, x_m]/I \otimes_k k[y_1, \ldots, y_n]/J \to k[x_1, \ldots, x_m, y_1, \ldots, y_n]/(R \cdot I + R \cdot J)
$$

is clearly an isomorphism, and we know that the left hand side is a domain so $R \cdot I + R \cdot J = \sqrt{R \cdot I + R \cdot J}$ and we win. □

\(^1\)Here we use that $k$ is algebraically closed. In fact, the statement is not true without this assumption. Consider for example $\mathbb{C} \otimes_k \mathbb{C}$!