Quantum Entanglement and Quantum Marginal Problem

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Abstract
In this paper we investigate relations between spectra of reduced density matrices and the spectrum of the full density matrix. This is known in the literature as the quantum marginal problem or the question of compatibility of local spectra. We provide a survey of known results for this problem and give an alternative derivation for one of these results. We also introduce a new technique, in the context of an example, that can be used to yield inequality relationships among entropy of some density matrices. Furthermore, we investigate the entropy and time evolution of the reduced density matrix of the eigenvector of a specific group of Hamiltonians after they are subjected to a perturbation. Finally, we determine how the information spreads in a system of $n$ spins placed in a circle subjected to a Hamiltonian that only couples consecutive spins.

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1 Introduction

It is a postulate of quantum mechanics that any isolated physical system can be associated with a Hilbert Space known as state space of the system. The system is completely described by its state vector, which is a unit vector in the system’s state space. The simplest quantum mechanical system is qubit. A qubit is a two-level state space. Suppose $|0\rangle$ and $|1\rangle$ form an orthonormal basis for this state space. Then, every state vector in this state space can be written as $|\psi\rangle = a|0\rangle + b|1\rangle$, with $|a|^2 + |b|^2 = 1$ because we have to have $\langle \psi | \psi \rangle = 1$. Two state spaces can be combined through tensor product: let $V$ and $W$ denote state spaces of system $A$ and $B$, respectively. Then, $V \otimes W$ is the state space of composite system $AB$.

A quantum system whose state $|\psi\rangle$ is known exactly is said to be pure state. In this case the density matrix which we call pure density matrix, is $\rho = |\psi\rangle \langle \psi|$. Otherwise, $\rho$ is called mixed state (or density matrix). In general, a mixed state $\rho$ can be written as $\rho = \sum_{i=1}^{n} p_i |\psi_i\rangle \langle \psi_i|$, where $n > 1$, $|\psi_i\rangle$’s are distinct and $\sum_{i=1}^{n} p_i = 1$. This formula can be interpreted in the following way: the probability of the system being in state $|\psi_i\rangle$ is $p_i$. For example, consider a system of two spins that is in state $|\uparrow \downarrow\rangle$ with probability $\frac{2}{3}$ and state $\frac{1}{\sqrt{2}} |\uparrow \uparrow\rangle + \frac{1}{\sqrt{2}} |\downarrow \downarrow\rangle$ with probability $\frac{1}{3}$. The mixed density matrix corresponding to this system is,

$$\rho = \frac{2}{3} |\uparrow \downarrow\rangle \langle \uparrow \downarrow| + \frac{1}{6} |\uparrow \uparrow\rangle \langle \uparrow \uparrow| + \frac{1}{6} |\uparrow \downarrow\rangle \langle \downarrow \uparrow| + \frac{1}{6} |\downarrow \uparrow\rangle \langle \uparrow \downarrow|.$$

Suppose a quantum system consists of two parts, $A$ and $B$, and let $\rho_{AB}$ be a density matrix on the composite system $AB$. The states $\rho_A$ and $\rho_B$ obtained by tracing out the subsystems $B$ and $A$, respectively, are called one-party reduced density matrices of $\rho_{AB}$. For instance, in the above example, let the first and the second spin be the system $A$ and $B$, respectively. Then,

$$\rho_A = \frac{5}{6} |\uparrow\rangle \langle \uparrow| + \frac{1}{6} |\uparrow\rangle \langle \downarrow| + \frac{1}{6} |\downarrow\rangle \langle \uparrow| + \frac{1}{6} |\downarrow\rangle \langle \downarrow|,$$

In the literature sometimes the terms “density operator”, “density matrix”, “state” and “state vector” are used interchangeably. In this paper, we follow this practice whenever it does not cause confusion. One needs to verify that the above definitions are well defined; that is, for pure state $\rho$ we cannot have $\rho = \sum p_i \rho_i$ with positive $p_i$ and distinct pure states $\rho_i$. This is done in the context of corollary 2.7.
\[
\rho_B = \frac{2}{3} | \downarrow \rangle \langle \downarrow | + \frac{1}{3} | \uparrow \rangle \langle \uparrow |.
\]

The reduced states are subject to some obvious compatibility conditions such as the consistency conditions

\[
\text{tr}_Q(\rho_{P \cup Q}) = \text{tr}_R(\rho_{P \cup R}),
\]

for any subsets \( P, Q, R \) such that \( P \) and \( Q \) are disjoint and \( P \) and \( R \) are disjoint. In this paper, we use \( \text{tr}_Q \) to denote tracing out subsystem \( Q \). We also use \( \text{tr}_\bar{Q} \) to denote tracing out all subsystems except subsystem \( Q \). The \( m \)-party reduced density matrix of a composite system of \( n \) parts is the generalization of the above definition in the obvious way.

The set of the eigenvalues of the density matrix carries important information about the system. For example, a system is in pure state if and only if one of its eigenvalues is one and the rest are zero. The quantum marginal problem (also known as question of compatibility of local spectra) is about relations between spectra of reduced states and the spectrum of the full state. In recent years, this problem has been particularly studied for one-party reduced states of the full state. In subsection 2.1 we present a survey of known results.

An important variant of quantum marginal problem dealing with a system of \( N \) identical fermions is known as \( N \)-representability problem. In 1995 it was designated by National Research Council of USA as one of the ten most prominent research challenges in quantum chemistry. Klyachko’s results regarding compatibility conditions of one-party reduced state [5] can be used to give a general solution of \( N \)-representability problem for one-particle reduced state. However, the necessary and sufficient conditions for the two-party reduced state are still unknown; that is, the complete set of relations between spectrum of state \( \rho_{ABC} \) and the spectra of its reduced states \( \rho_{AB}, \rho_{AC} \) and \( \rho_{BC} \) are not known. This problem has been of great interest in the field of quantum chemistry, since knowing the conditions on the two-party states would allow the calculation of the ground state energy of an \( N \)-fermion Hamiltonian involving 2-body interactions only from the 2-party state and an effective 2-body Hamiltonian. This is still a topic of active research (see for example [11, 10]).

As we will see in section 4, the quantum marginal problem is related to the notion of quantum entanglement. Quantum entanglement is a property of a quantum state of a composite system in which the quantum states of the constituting objects are linked together so that one object can no longer be adequately described without full mention of its counterpart. Entanglement is the property that separates quantum systems from classical systems; it is a uniquely quantum mechanical resource that plays a key role in many of the most interesting applications of quantum computation, quantum information and many other fields. For example, quantum entanglement has become a subject of interest for string theorists in recent years because it provides some insights into the AdS/CFT correspondence [3]. They are known examples of AdS/CFT correspondence in

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\(^3\text{Recall that the spectrum of a matrix is the set of its eigenvalues.}\)

\(^4\text{AdS/CFT correspondence (also known as gauge-theory/gravity duality) is the conjecture that certain quantum field theory is actually equivalent to some quantum gravity systems.}\)
which entanglement corresponds to geometrical property in quantum gravity (see [3] for more information).

One way of quantifying the entanglement between degrees of freedom in one subsystem is through von Neumann entropy (or simply entropy for conciseness). For a density matrix $\rho$, its entropy is defined as

$$S(\rho) \equiv -\text{tr}(\rho \log \rho).$$

In the above formula and rest of this paper all logarithms are taken in base 2. If eigenvalues of $\rho$ are $\lambda_i$’s, for $1 \leq i \leq n$, then the above formula can be rewritten in the form,

$$S(\rho) = H(\lambda_1, \ldots, \lambda_n) \equiv - \sum_{i=1}^{n} \lambda_i \log \lambda_i,$$

where $H$ is the entropy function. Here, by convention, we have $0 \log 0 \equiv 0$. Entropy has many interesting properties. Some of these properties are discussed in subsection 2.2.

Entropy can be viewed as a measure of entanglement: If $\rho_{AB}$ is density matrix of system $AB$, $S(\rho_A)$ is a measure of how entangled subsystem $A$ is with the rest of the system (i.e. subsystem $B$). Incidentally, $S(\rho_A) = 0$ would indicate that subsystem $A$ is not entangled with the rest of the system.

In the next section we define some of the notations that we will use in the rest of the paper, provide a survey of some known results for the quantum marginal problem, and describe some of the most important properties of entropy in theorem 2.6. In section 3 we provide a different proof for theorem 2.3 which was first discovered by Higuchi an others in 2003. In section 4 we use Schur-concavity of the entropy function and some linear algebraic properties of the density matrix to establish theorem 4.4. We hope that the techniques applied in section 4 can be used to establish new inequalities among entropy of different density matrices. In section 5 we investigate the entropy and time evolution of the reduced density matrix of the eigenvector of a good Hamiltonian after it is subjected to a perturbation. In section 6 we determine how the information spreads in a system of $n$ spins placed in a circle subject to a Hamiltonian that only couples consecutive spins.

## 2 Theory and some known results

We start this section by describing notations and definitions that are used in the subsequent sections. In subsections 2.1 and 2.2 we make a survey of some known results about quantum marginal problem and properties of entropy.

A spin is an example of a qubit since it is a two-level state space. A spin has exactly two basis: $|\downarrow\rangle$, which we call spin down, and $|\uparrow\rangle$, which we call spin up. We sometime use $|0\rangle$ and $|1\rangle$ to denote state vectors $|\downarrow\rangle$ and $|\uparrow\rangle$, respectively. The general state of a spin is therefore $a|0\rangle + b|1\rangle$, with $|a|^2 + |b|^2 = 1$. Consequently, a state vector of a system of $n$ spins corresponds to a linear combination of sequences of $n$ zeroes and ones. To avoid cumbersome notation, we sometimes view these sequences as a binary number and use their decimal representation to denote them. For example, in a system of three spins we can use either $\frac{1}{\sqrt{2}}|001\rangle + \frac{1}{\sqrt{2}}|101\rangle$ or $\frac{1}{\sqrt{2}}|1\rangle + \frac{1}{\sqrt{2}}|5\rangle$ to denote state vector $\frac{1}{\sqrt{2}}|\downarrow\downarrow\uparrow\rangle + \frac{1}{\sqrt{2}}|\uparrow\downarrow\uparrow\rangle$. 
We can act on a spin using spin operators. Some of the most famous spin operators that we will use in this paper are $\sigma_z$, $\sigma_+$, and $\sigma_-$. These operators act on state vector of a spin according to the following rules:

$$
\sigma_z|0\rangle = -|0\rangle, \quad \sigma_z|1\rangle = |1\rangle, \quad \sigma_+|0\rangle = 2|1\rangle, \quad \sigma_+|1\rangle = 0, \quad \sigma_-|0\rangle = 0, \quad \sigma_-|1\rangle = 2|0\rangle.
$$

We can combine operators acting on different spins using tensor product to form an operator acting on a system of $n$ spins. For example, $\sigma_z^1 \otimes \sigma_z^2$ is an operator that acts on a system of two spins with $\sigma_z$ operating on the first spin and $\sigma_z$ operating on the second spin. Sometimes we use the shorthand notation $\sigma^r_{+1} \ldots \sigma^d_{+1} \otimes \ldots \otimes \sigma^r_{+d} \otimes \ldots \otimes I^{n+m}$ (here each * can be replaced by any of $z$, $+$, or $-$ and $I$ denotes the identity operator).

In section 5, we consider a specific group of Hamiltonians, which are defined in the next definition.

**Definition 2.1** Let a composite system be comprised of $n$-level quantum subsystem $A$ and $m$-level quantum subsystem $B$. We call Hamiltonian $H$ good if it is of the form

$$
H = \sum_{i=1}^{n} \sum_{j=1}^{m} \lambda_{ij} |i^A\rangle \langle j^B|,
$$

where $\lambda_{ij}$’s are arbitrary real numbers$^5$.

The following decomposition is a ubiquitous tool in the study of composite systems. We state it here for the sake of easy reference.

**Theorem 2.2 (Schmidt decomposition, [6, theorem 2.7] )** Suppose $|\psi\rangle$ is a pure state of a composite system, $AB$. Then, there exist orthonormal states $|i^A\rangle$ for system $A$, and orthonormal states $|i^B\rangle$ of system $B$ such that

$$
|\psi\rangle = \sum_i \lambda_i |i^A\rangle |i^B\rangle,
$$

where $\lambda_i$ are nonnegative real numbers satisfying $\sum_i \lambda_i^2 = 1$.

### 2.1 Quantum marginal problem

Here are some known results related to the quantum marginal problem. All of them have been discovered recently.

**Theorem 2.3 (Higuchi et al. [11], 2003)** Suppose $\rho_i$ (for $1 \leq i \leq n$) be a set of $n$ one-qubit density operators. Let $\lambda_i$ denote the minimal eigenvalue of $\rho_i$. Then, a necessary and sufficient condition for $\rho_i$’s to be the reduced states of a pure $n$-qubit density operator is that $\lambda_i$’s satisfy polygonal inequalities. That is,

$$
\lambda_k \leq \lambda_1 + \cdots + \lambda_{k-1} + \lambda_{k+1} + \cdots + \lambda_n, \quad \text{for } k = 1, \ldots, n.
$$

$^5$In fact, $\lambda_{ij}$’s are the eigenvalues of $H$. 

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A necessary and sufficient condition for two-qubit systems of two-qubits.

In section 3 we give an alternative proof for the above theorem.

The following theorem provides the complete solution of quantum marginal problem for systems of two-qubits.

**Theorem 2.4 (Bravyi [4], 2004)** A necessary and sufficient condition for two-qubit density operator $\rho_{AB}$, with eigenvalues $\lambda_1^{AB} \geq \lambda_2^{AB} \geq \lambda_3^{AB} \geq \lambda_4^{AB}$, to have one-qubit reduced density operators $\rho_A$ and $\rho_B$ is given by the following inequalities:

$$\min(\lambda_A, \lambda_B) \geq \lambda_3^{AB} + \lambda_4^{AB},$$

$$\lambda_A + \lambda_B \geq \lambda_2^{AB} + \lambda_3^{AB} + 2\lambda_4^{AB},$$

$$|\lambda_A - \lambda_B| \leq \min(\lambda_1^{AB}, \lambda_2^{AB}, \lambda_3^{AB} - \lambda_4^{AB}),$$

where $\lambda_A$ and $\lambda_B$ are minimal eigenvalues of $\rho_A$ and $\rho_B$, respectively.

A *quitrit* is a three-level state space. By Schmidt decomposition, reduced states $\rho_A$ and $\rho_{BC}$ of pure state $\rho_{ABC}$ have the same nonzero eigenvalues. Consequently, the following theorem provides the complete solution of quantum marginal problem for pure states of three-quitrit systems.

**Theorem 2.5 (Higuchi [2], 2003)** A necessary and sufficient condition for nine nonnegative numbers $\lambda_i^a$, $i = 1, 2, 3$, $a = A, B, C$ satisfying $\lambda_1^a \leq \lambda_2^a \leq \lambda_3^a$ and $\lambda_1^a + \lambda_2^a + \lambda_3^a = 1$ to be the eigenvalues of the three one-quitrit reduced density matrices of a pure three-quitrit quantum state $\rho_{ABC}$ is given by the following inequalities:

$$\lambda_2^a + \lambda_1^a \leq \lambda_2^b + \lambda_1^b + \lambda_2^c + \lambda_1^c,$$

$$\lambda_3^a + \lambda_1^a \leq \lambda_2^b + \lambda_1^b + \lambda_2^c + \lambda_1^c,$$

$$\lambda_3^a + \lambda_2^a \leq \lambda_2^b + \lambda_1^b + \lambda_2^c + \lambda_1^c,$$

$$2\lambda_2^a + \lambda_1^a \leq 2\lambda_2^b + \lambda_1^b + 2\lambda_2^c + \lambda_1^c,$$

$$2\lambda_1^a + \lambda_2^a \leq 2\lambda_2^b + \lambda_1^b + 2\lambda_1^c + \lambda_2^c,$$

$$2\lambda_1^a + \lambda_3^a \leq 2\lambda_2^b + \lambda_1^b + 2\lambda_3^c + \lambda_2^c,$$

$$2\lambda_2^a + \lambda_3^a \leq 2\lambda_1^b + \lambda_2^b + 2\lambda_3^c + \lambda_2^c,$$

where $(abc)$ are any permutations of $(ABC)$.

In 2004, Klyachko solves the most general form of the quantum marginal problem for one-party reduced states in his celebrated work [5]. We do not state Klyachko’s result here as it is somewhat complicated and requires several definitions. Nevertheless, Klyachko gives a set of linear inequalities on spectra $\rho_{AB}$, $\rho_A$ and $\rho_B$ that are necessary and sufficient for the existence of state $\rho_{AB}$ and its reduced states $\rho_A$ and $\rho_B$. Furthermore, it turns out that there is a connection between the above problem and representation theory. To each spectrum one can associate a representation of the symmetric group defined by a Young diagram whose normalized row lengths approximate the spectrum. Klyachko [5] and Christandl et al. [12], separately, have shown that triple of spectra

$$(\text{Spec } \rho_A, \text{Spec } \rho_B, \text{Spec } \rho_{AB})$$

correspond to young diagram $(\mu, \nu, \lambda)$ with nonzero Kronecker coefficient $g_{\mu\nu\lambda}$. Christandl et al. [12] have also shown that these spectral triples form a convex polytope.
As stated in the introduction, despite successful development for one-party reduced state, the quantum marginal problem is still unsolved for two-party reduced states; that is, a necessary and sufficient conditions for density matrices $\rho_{AB}$, $\rho_{AC}$ and $\rho_{BC}$ to be two-party reduced state of $\rho_{ABC}$ is still unknown. In recent years, however, there has been some limited progress in answering this problem. For example, Han et al. \cite{9} have recently found some necessary conditions for this problem. This is still a topic of active research.

2.2 Entropy

Entropy has many interesting properties. Below we state some of them. For more information see for example \cite{6}.

**Theorem 2.6 \cite{6, sections 11.2-11.4}**

1. The entropy is nonnegative. The entropy is zero if and only if the state is pure.

2. In a $d$-dimensional Hilbert space, the entropy is at most $\log d$. The entropy is equal to $\log d$ if and only if the system is in completely mixed state $I/d$.

3. Suppose a composite system $AB$ is in a pure state. Then, $S(A) = S(B)$.

4. Given a set of nonnegative real numbers $p_i$ with $\sum_{i=1}^{n} p_i = 1$ and density operators $\rho_i$, we have

$$\sum_{i=1}^{n} p_i S(\rho_i) \leq S(\sum_{i=1}^{n} p_i \rho_i) \leq \sum_{i=1}^{n} p_i S(\rho_i) + H(p_1, \ldots, p_n).$$

The equality is attainable for both inequalities. In particular, the lower bound is achieved if and only if all states $\rho_i$ for which $p_i > 0$ are identical. Thus, entropy is a strictly concave function.

5. **(Additivity of entropy)** Given two density operators $\rho_A$ and $\rho_B$ describing independent systems $A$ and $B$, we have

$$S(\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B).$$

6. **(Subadditivity)** Let $\rho_A$ and $\rho_B$ be reduced density operators of state $\rho_{AB}$. Then,

$$|S(\rho_A) - S(\rho_B)| \leq S(\rho_{AB}) \leq S(\rho_A) + S(\rho_B).$$

7. **(Strong subadditivity)** Let $\rho_B$, $\rho_{AB}$ and $\rho_{BC}$ be reduced density operators of state $\rho_{ABC}$. Then,

$$S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC}).$$

In section \cite{4} we derive another inequality regarding entropy using linear algebraic methods. As an application of the above theorem, we prove the following corollary.
Corollary 2.7 For pure density operators $\rho$ and $\rho_i$, $1 \leq i \leq n$, we can have $\rho = \sum_i p_i \rho_i$, for positive $p_i$’s with $\sum_i p_i = 1$, if and only if $\rho = \rho_1 = \ldots = \rho_n$.

Proof: We have

$$0 = S(\rho) = S\left(\sum_i p_i \rho_i\right) \geq \sum_i p_i S(\rho_i) = 0,$$

where the first and the last equalities are due to the fact that $\rho$, $\rho_i$’s are pure (theorem 2.6 part 1) and the inequality is due to part 4 of theorem 2.6. Subsequently, we have to have equality in equation (2). This implies (using the equality condition of part 4 of theorem 2.6) that $\rho_i$’s are identical. This follows the result.

3 Alternative proof for Higuchi et al. [1]

In this section, we present an alternative proof for theorem 2.3. In section 3.1 we first find some relations between spectra of one-party reduced states and spectrum of bi-partite system (Theorem 3.1). We then use this result to prove the necessity of condition (1). In section 3.2, we show that for any set of nonnegative numbers satisfying (1), we can find a pure state for which the minimum eigenvalue of its one-qubit reduced density matrices are the given numbers. This proves the sufficiency of condition (1).

3.1 Necessity of polygonal inequalities

Let $\rho_{AB}$ be a state of composite system $AB$, where $A$ and $B$ have dimensions $m$ and $p$, respectively. By spectral decomposition there is an orthonormal set of states $|\Phi_{iAB}\rangle$’s in space $AB$, such that

$$\rho_{AB} = \sum_{i=1}^{mp} \lambda_{iAB} |\Phi_{iAB}\rangle \langle \Phi_{iAB}|.\quad (3)$$

Observe that $\lambda_{iAB}$’s are eigenvalues of $\rho_{AB}$.

Let $|\varphi_j^A\rangle$ and $|\varphi_k^B\rangle$ be, respectively, the eigenvectors of density operators $\rho_A$ and $\rho_B$. Consequently, state vectors $|\varphi_j^A\rangle$ (for $1 \leq j \leq m$) form an orthonormal set of basis for space $A$. Similarly, state vectors $|\varphi_k^B\rangle$ (for $1 \leq k \leq p$) form an orthonormal set of basis for space $B$. Consequently, state vectors $|\varphi_j^A\rangle |\varphi_k^B\rangle$ (for $1 \leq j \leq m$ and $1 \leq k \leq p$) form an orthonormal set of basis for space $AB$. Thus, there exist complex numbers $a_{jk}^i$ such that

$$|\Phi_{iAB}\rangle = \sum_{j,k} a_{jk}^i |\varphi_j^A\rangle |\varphi_k^B\rangle,\quad (4)$$

for $1 \leq i \leq mp$. Observe that since states $|\Phi_{iAB}\rangle$ are orthonormal, we must have that

$$\sum_{j,k} (a_{jk}^i)^* a_{jk}^{i'} = \delta_{ii'}.$$

Spectral decomposition states that an operator is diagonalizable if and only if it is normal [6, theorem 2.1]. Note that every density operator is hermitian and therefore normal.
Substituting (4) into equation (3), we have

\[ \rho_{AB} = \sum_{i,j,k,j',k'} \lambda^A_{ij} \lambda^B_{jk} (a^i_j a^k_{j'})^* |\varphi_j^A \varphi_k^B \rangle \langle \varphi_j^A \varphi_k^B| \]

Since \(|\varphi_j^A\rangle\) and \(|\varphi_k^B\rangle\) are, respectively, the eigenvectors of density operators \(\rho_A\) and \(\rho_B\), it is easy to deduce from the above equation that

\[ \begin{align*}
\lambda^A_j &= \sum_{i,k} \lambda^A_{ij} |a^i_{jk}|^2 \\
\lambda^B_k &= \sum_{i,j} \lambda^B_{ij} |a^i_{jk}|^2
\end{align*} \]

where \(\lambda^A_j\) and \(\lambda^B_k\) are, respectively, eigenvalues of density operators \(\rho_A\) and \(\rho_B\). By relabeling (if necessary) of \(j\), \(k\), and \(i\), we may assume that \(\lambda^A_j\), \(\lambda^B_k\), and \(\lambda^A_{ij}\) decrease as \(j\), \(k\), and \(i\) increase, respectively.

**Theorem 3.1** Let the notations be defined as above. Assume \(r\) and \(s\) are two natural numbers such that \(1 \leq r \leq m\) and \(1 \leq s \leq p\). Then

\[ 1 - \sum_{j=1}^{r} \lambda^A_j + 1 - \sum_{k=1}^{s} \lambda^B_k \geq 1 - \sum_{i=1}^{rs} \lambda^A_{ij}. \]

(6)

To prove the theorem we first show the following two lemmas:

**Lemma 3.2** Let \(a^i_{jk}\) be given as above. Then

\[ \sum_{i=1}^{mp} |a^i_{jk}|^2 = 1. \]

**Proof:** Let \(\vec{v}_i\), for \(1 \leq i \leq mp\), be a column with \(mp\) entries whose entries are \(a^i_{jk}\). By (5) we know that vectors \(\vec{v}_i\), for \(1 \leq i \leq mp\), are orthonormal. Consequently, matrix \(U = [\vec{v}_1 \vec{v}_2 \ldots \vec{v}_{mp}]\) is unitary. Therefore

\[ UU^\dagger = I. \]

Comparing the diagonal entries of the both side of the former expression yields the result. \(\blacksquare\)

**Lemma 3.3** Let \(A_1 \geq A_2 \cdots \geq A_n\), \(S\), and \(y_i\) (for \(1 \leq i \leq n\)) be nonnegative real numbers. Then, the maximum of

\[ A_1 x_1 + \cdots + A_n x_n \]

(7)

where

(i) \(x_i \in [0, y_i]\), for \(1 \leq i \leq n\), and

(ii) \(\sum_{i=1}^{n} x_i = S\),

is achieved when \((x_1, \ldots, x_n) = (y_1, \ldots, y_j, t, 0, \ldots, 0)\) (here \(t\) and index \(j\) are chosen such that \(t = S - \sum_{i=1}^{j} y_i\) and \(t \leq y_{j+1}\)).
Proof: Let \((x_1, \ldots, x_n)\) be any n-tuple that satisfies conditions (i) and (ii). We perform the following operation: if for two indices \(i\) and \(j\) with \(i < j\), \(x_i < y_i\) and \(0 < x_j\) then replace \(x_i\) and \(x_j\) with either \(y_i\) and \(x_j - (y_i - x_i)\) or \(x_i + x_j\) and \(0\) depending whichever case would yield two nonnegative numbers (note that one of them has to do so). By performing this operation the value of \([7]\) would increase or at least would not change. Performing this operation repeatedly would change \((x_1, \ldots, x_n)\) into \((y_1, \ldots, y_j, t, 0, \ldots, 0)\). This follows the result.

Proof of theorem 3.1: Since the eigenvalues of system \(A\) (respectively \(B\)) add up to 1, we have

\[
1 - \sum_{j=1}^{r} \lambda^A_j + 1 - \sum_{k=1}^{s} \lambda^B_k = \sum_{j=r+1}^{m} \lambda^A_j + \sum_{k=s+1}^{p} \lambda^B_k = \sum_{i,j} \lambda^{AB}_i |a^i_{jk}|^2 + \sum_{k=s+1}^{p} \lambda^B_k |a^i_{jk}|^2 = \sum_{i,j} \lambda^{AB}_i \left( \sum_{j=r+1}^{m} |a^i_{jk}|^2 + \sum_{j=s+1}^{p} |a^i_{jk}|^2 \right) \geq \sum_{i,j} \lambda^{AB}_i \left( \sum_{j=r+1}^{m} |a^i_{jk}|^2 + \sum_{j=s+1}^{p} |a^i_{jk}|^2 \right) = \sum_{i,j} \lambda^{AB}_i \left( 1 - \sum_{j=r+1}^{m} |a^i_{jk}|^2 \right)
\]

where we used \([5]\) (with \(i' = i\)) for the fifth and the fact that the sum of eigenvalues of system \(AB\) is 1 for the sixth equality.

Because of \([5]\), for every \(i\),

\[
\sum_{j=1, k=1}^{j=r, k=s} |a^i_{jk}|^2 \leq \sum_{j=1, k=1}^{j=m, k=p} |a^i_{jk}|^2 = 1.
\]

Furthermore, by lemma 3.2

\[
\sum_{i} \sum_{j=1, k=1}^{j=r, k=s} |a^i_{jk}|^2 = \sum_{j=1, k=1}^{j=r, k=s} \sum_{i} |a^i_{jk}|^2 = rs.
\]

Thus, by lemma 3.3 we can conclude that

\[
\sum_{i} \lambda^{AB}_i \left( \sum_{j=1, k=1}^{j=r, k=s} |a^i_{jk}|^2 \right) \leq \sum_{i=1}^{rs} \lambda^{AB}_i.
\]

Plugging the above into \([8]\), yields \([6]\).

Applying theorem 3.1 repeatedly would result into the following Corollary:

Corollary 3.4 Assume \(\rho_{A_1}, \ldots, \rho_{A_n}\) are one-party reduced states of full state \(\rho_{A_1 \cdots A_n}\). For every state \(\rho_X\) of system \(X\), let \(|X|\) denote dimension of the space and \(\lambda^X_1, \ldots, \lambda^X_{|X|}\) be eigenvalues of \(\rho_X\) sorted in non-increasing order. Then,

\[
1 - \sum_{i=1}^{r_1} \lambda^A_1 + 1 - \sum_{i=1}^{r_2} \lambda^A_2 + \cdots + 1 - \sum_{i=1}^{r_n} \lambda^A_n \geq 1 - \sum_{i=1}^{r_1 r_2 \cdots r_n} \lambda^{A_1 A_2 \cdots A_n}_i,
\]

where \(r_j \leq |A_j|\) for \(1 \leq j \leq n\).
Now we prove the necessity of condition (1). Because of symmetry we only need to show the inequality holds for \( k = n \). Suppose we have a pure state \( \rho \) in system \( A_1 \cdots A_n \), where each \( A_i \) is a qubit. Observe that for every one-qubit reduced state \( \rho_{A_i} \) we only have two eigenvalues \( \lambda_{A_i}^1 \geq \lambda_{A_i}^2 \). Applying corollary 3.4 to \((n-1)\)-qubit reduced state \( \rho_{A_1 \cdots A_{n-1}} \) with \( r_1 = r_2 = \cdots = r_{n-1} = 1 \) and using the fact that \( \lambda_{A_i}^2 = 1 - \lambda_{A_i}^1 \) for each \( A_i \) yields,

\[
\lambda_{A_1}^2 + \lambda_{A_2}^2 + \cdots + \lambda_{A_{n-1}}^2 \geq 1 - \lambda_{A_1 \cdots A_{n-1}}^1.
\]

(9)

Since \( \rho \) is a pure state, by Schmidt decomposition, reduced states \( \rho_{A_1 \cdots A_{n-1}} \) and \( \rho_{A_n} \) have the same nonzero eigenvalue. Therefore,

\[
1 - \lambda_{A_1 \cdots A_{n-1}}^1 = 1 - \lambda_{A_n}^1 = \lambda_{A_n}^2.
\]

Substituting the above into inequality (9), yields the necessity of condition (1) for \( k = n \).

### 3.2 Sufficiency of polygonal inequalities

In this subsection, we show that the equality is attainable for condition (1). Consider a system on \( n \) qubits (spins) labeled 1 to \( n \). Let \( |\psi_{ij} \rangle \) be the state vector whose \( i \)-th and \( j \)-th qubit are spin down while the remaining of the qubit are spin up. Also, let \( |\psi_0 \rangle \) be the state vector whose qubits are all spin down. It is clear from the definition that \( |\psi_{ij} \rangle = |\psi_{ji} \rangle \). Let \( a_0 \) and \( a_{ij} = a_{ji} \), for \( 1 \leq i < j \leq n \), be nonnegative real numbers, with \( a_0^2 = 1 - \sum_{i<j} a_{ij}^2 \). Set

\[
|\psi \rangle = a_0 |\psi_0 \rangle + \frac{1}{2} \sum_{i \neq j} a_{ij} |\psi_{ij} \rangle = a_0 |\psi_0 \rangle + \sum_{i<j} a_{ij} |\psi_{ij} \rangle.
\]

(10)

Observe that \( |\psi \rangle \) is normalized due to the choice of \( a_0 \). Recall that we use \( \text{tr}_r \) to denote tracing out system of all qubits except the \( r \)-th qubit. If \( \{i, j\} \neq \{k, l\} \) then for every index \( 1 \leq r \leq n \)

\[
\text{tr}_r |\psi_{ij} \rangle \langle \psi_{kl}| = 0,
\]

whereas

\[
\text{tr}_r |\psi_{ij} \rangle \langle \psi_{ij}| = \begin{cases} 
| \uparrow \rangle \langle \uparrow | & \text{for } r \notin \{i, j\} \\
| \downarrow \rangle \langle \downarrow | & \text{for } r \in \{i, j\}
\end{cases}
\]

Also it is clear that

\[
\text{tr}_r |\psi_0 \rangle \langle \psi_0| = | \uparrow \rangle \langle \uparrow |.
\]

Putting all these together, the one-qubit reduced state of the \( r \)-th spin is equal to,

\[
\rho_r = \text{tr}_r |\psi \rangle \langle \psi| = \left( \sum_{i \neq r} a_{ir}^2 \right) | \downarrow \rangle \langle \downarrow | + \left( a_0^2 + \sum_{i \neq r, j \neq r, i < j} a_{ij}^2 \right) | \uparrow \rangle \langle \uparrow |.
\]

(11)

Equation (11) implies that \( \sum_{i \neq r} a_{ir}^2 \) is an eigenvalue of \( \rho_r \). Ultimately, we will show that given \( \lambda_1, \ldots, \lambda_n \), there exist a choice of \( a_{ij} \)’s such that for \( r = 1, \ldots, n \) the system of equations

\[
\sum_{i \neq r} a_{ir}^2 = \lambda_r
\]

(12)
are satisfied if and only if we have $\{1\}$. It is straightforward that if there exist a choice of $a_{ij}$‘s satisfying $\{1\}$, then we have $\{1\}$. We need to show that $\{1\}$ is also a sufficient condition. Observe that it suffices to show that given $\{1\}$, there exist an antisymmetric matrix $A$ (whose entries are $a_{ij}$) such that the diagonal entries of $A^2$ are $-\lambda_1, \ldots, -\lambda_n$. This is done in theorem $3.6$. First we prove the following lemma.

**Lemma 3.5** Let $\alpha_i, 1 \leq i \leq n$, be real numbers from the interval $[0, 1]$ with $\sum_{i=1}^n \alpha_i = 2$. Then, there exist orthonormal real vectors $\vec{u} = (u_1, \ldots, u_n)$ and $\vec{v} = (v_1, \ldots, v_n)$ such that

$$(u_1^2 + v_1^2, \ldots, u_n^2 + v_n^2) = (\alpha_1, \ldots, \alpha_n).$$

**Proof:** We may assume, by relabeling if necessary, that $\alpha_1 \leq \cdots \leq \alpha_n$. Set $v_n = -\sqrt{\alpha_n - u_n^2}$ and $v_i = \sqrt{\alpha_i - u_i^2}$ for $i = 1, \ldots, n - 1$. We call a set of nonnegative real numbers $u_1, \ldots, u_n$ “proper” if $u_i^2 \leq \alpha_i$, for $1 \leq i \leq n$, and $u_1^2 + \cdots + u_n^2 = 1$. We, therefore, are looking for a proper set of $u_1, \ldots, u_n$ such that $f(u_1, \ldots, u_n) = 0$ where

$$f(u_1, \ldots, u_n) = -u_n \sqrt{\alpha_n - u_n^2} + \sum_{i=1}^{n-1} u_i \sqrt{\alpha_i - u_i^2}.$$

We indirectly prove the existence of such a proper set of $u_1, \ldots, u_n$. First, we find a proper set of $u_1, \ldots, u_n$ for which $f(u_1, \ldots, u_n) \geq 0$. We also find another proper set of $u_1, \ldots, u_n$ for which $f(u_1, \ldots, u_n) \leq 0$. Then, by continuity, there must exist a proper set of $u_1, \ldots, u_n$ for which $f(u_1, \ldots, u_n) = 0$.

There exist $j < n$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_j \geq 1$. Choose smallest such an index $j$ (i.e. $\alpha_1 + \alpha_2 + \cdots + \alpha_{j-1} \leq 1$); this choice is possible due to the given assumptions for $\alpha_i$’s. Set

$$u_i = \begin{cases} \sqrt{\alpha_i} & \text{for } i < j \\ \sqrt{1 - \alpha_1 - \cdots - \alpha_{j-1}} & \text{for } i = j \\ 0 & \text{for } i > j \end{cases}$$

It is easy to verify that the above choice of $u_i$‘s makes a proper set with $f(u_1, \ldots, u_n) \geq 0$.

Notice that the maximum of function $x \sqrt{\alpha - x^2}$ is $\alpha/2$, which is attained for $x = \sqrt{\alpha/2}$. Again, due to the given assumptions for $\alpha_i$’s, there exist $j < n$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_j \geq 1 - \alpha_n/2$. Choose smallest such an index $j$ (i.e. $\alpha_1 + \alpha_2 + \cdots + \alpha_{j-1} \leq 1 - \alpha_n/2$ ). Set

$$u_i = \begin{cases} \sqrt{\alpha_i} & \text{for } i < j \\ \sqrt{1 - \alpha_n/2 - \alpha_1 - \cdots - \alpha_{j-1}} & \text{for } i = j \\ \sqrt{\alpha_n/2} & \text{for } j < i < n \\ 0 & \text{for } i = n \end{cases}$$

It is easy to verify that the above choice of $u_i$‘s makes a proper set. Furthermore, for this choice of $u_i$‘s,

$$f(u_1, \ldots, u_n) = -\alpha_n/2 + u_j \sqrt{\alpha_j - u_j^2} \leq -\alpha_n/2, \quad \alpha_j \leq \alpha_n/2 \leq 0.$$

This follows the result.
Theorem 3.6 Given that $\lambda_1, \ldots, \lambda_n$ satisfy (1), there exist an antisymmetric matrix $A$ such that the diagonal entries of $A^2$ are $-\lambda_1,\ldots,-\lambda_n$.

Proof: Let $\gamma^2 = \frac{1}{2} \sum_{j=1}^n \lambda_j$ and define $\alpha_i = \frac{\lambda_i}{\gamma}$. Notice that because of (1), $\alpha_i$'s satisfy the constraints given in the Lemma 3.5. Thus by Lemma 3.5, there exist orthonormal real vectors $\vec{q}_1 = (q_{11}, q_{21}, \ldots, q_{n1})$ and $\vec{q}_2 = (q_{12}, q_{22}, \ldots, q_{n2})$ such that

$$
(q_{11}^2 + q_{12}^2, q_{21}^2 + q_{22}^2, \ldots, q_{n1}^2 + q_{n2}^2) = (\alpha_1, \alpha_2, \ldots, \alpha_n). \quad (14)
$$

We can extend vectors $\vec{q}_1$ and $\vec{q}_2$ to form orthonormal basis $\vec{q}_1, \ldots, \vec{q}_n$. Form orthogonal matrix $Q$ by appending vectors $\vec{q}_1, \ldots, \vec{q}_n$ side by side such that the $j$-th column of $Q$ is $\vec{q}_j$.

Now set

$$
A = Q \begin{bmatrix}
0 & \gamma & 0 & \ldots & 0 \\
-\gamma & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix} Q^T.
$$

We claim that $A$ is the desired matrix. First observe that $A$ is antisymmetric and

$$
A^2 = Q \begin{bmatrix}
-\gamma^2 & 0 & 0 & \ldots & 0 \\
0 & -\gamma^2 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{bmatrix} Q^T.
$$

It is easy to deduce that the $ii$-entry of $A^2$ is $-(q_{i1}^2 + q_{i2}^2)\gamma^2$. Recall that $(q_{i1}^2 + q_{i2}^2)\gamma^2 = \lambda_i$ by (14). Hence the $ii$-entry of $A^2$ is $-\lambda_i$. This follows the result. \hfill \ensuremath{\blacksquare}

We conclude this section by noting that we have basically proven the next theorem in graph theory:

**Theorem 3.7** Consider a complete nonnegative-weighted graph of $n$ vertices; that is, we assign positive real number $a_{ij}^2 \equiv a_{ji}^2$ to the edge connecting vertices $i$ and $j$, for all $1 \leq i < j \leq n$. The weight of vertex $i$, denoted by $\lambda_i$, is defined as the sum of the weights of all the edges incident to vertex $i$ (i.e. $\lambda_i = \sum_{r \neq i} a_{ir}^2$). Then, given a set of nonnegative real numbers $\lambda_i$, for $1 \leq i \leq n$, the necessary and sufficient conditions for the existence of a complete nonnegative-weighted graph of $n$ vertices are equations (1).

### 4 Entanglement inequality

Recall that if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $n$ dimensional density matrix $\rho$, then

$$
S(\rho) = H(\lambda_1, \ldots, \lambda_n).
$$

Furthermore, eigenvalues of every density matrix are nonnegative real numbers that add up to one. Consequently, $H$ can be thought as a function from $\mathbb{R}^{n-1}$ to $\mathbb{R}$, which takes in the
spectrum of the density matrix and maps it to its entropy. In Figure 1, the entropy function is depicted for the case $n = 2$. Entropy function has some very interesting properties, which can be used to establish restrictions on entropy of density matrices. In this section, using Schur-concavity of $H$ and some linear algebraic properties of $\rho$, we prove theorem 4.4. An equivalent of this theorem can be found in [6, theorem 11.9]; however, the proofs are quite distinct. We first start with some definitions and known results.

**Definition 4.1** Let $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ be real vectors. We say $y$ majorizes $x$, denoted by $x \prec y$, if the sum of $k$ largest entries of $y$ is greater or equal than the sum of $k$ largest entries of $x$ for $k = 1, \ldots, n - 1$, and the two sums are equal for $k = n$.

The following theorem is known as **Schur-Horn Theorem**. It originally appeared in [7, theorem 5]

**Theorem 4.2 (Schur-Horn)** There exist a hermitian matrix with diagonal entries $d_1, \ldots, d_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$, if and only if

$$(d_1, \ldots, d_n) \prec (\lambda_1, \ldots, \lambda_n).$$

**Definition 4.3** A function $f : \mathbb{R}^n \to \mathbb{R}$ is Schur-concave if $f(x) \geq f(y)$ whenever $x \prec y$.

It is well known that the entropy function $H(x_1, \ldots, x_n)$ is Schur-concave (see for example [8]).

Now we state and proof the main theorem of this section
**Theorem 4.4** Let $\rho$ be a density matrix of size $n$. Assume, $|v_i\rangle$, for $1 \leq i \leq n$, is a set of orthonormal basis. Then,

$$S(\rho) \leq S \left( \sum_{i=1}^{n} \langle v_i|\rho|v_i \rangle |v_i\rangle \langle v_i| \right).$$

(15)

**Proof:** First, we show that equation (15) is well defined; that is, matrix

$$\sum_{i=1}^{n} \langle v_i|\rho|v_i \rangle |v_i\rangle \langle v_i|$$

is a density matrix. To that end, we just need to show that trace of matrix (16) is equal to

$$\sum_{i=1}^{n} \langle v_i|\rho|v_i \rangle = \sum_{i=1}^{n} \text{tr}(\rho|v_i\rangle \langle v_i|) = \text{tr}(\rho \sum_{i=1}^{n} |v_i\rangle \langle v_i|) = \text{tr}(\rho I) = 1.$$

For matrix $A$, let $\lambda(A)$ and $d(A)$ denote vectors whose entries, which are sorted in non-increasing order, are the eigenvalues and diagonal entries of matrix $A$, respectively. Form unitary matrix $U$ by appending vectors $|v_i\rangle$ side by side such that the $j$-th column of $U$ is $|v_j\rangle$.

Observe that $|v_i\rangle = U|e_i\rangle$ where $|e_i\rangle$’s are the standard basis. Define

$$\rho' = U^\dagger \rho U.$$

Then,

$$S(\rho) = H(\lambda(\rho)) = H(\lambda(\rho')),$$

(17)

where the second equality is due to the fact that $\rho$ and $\rho'$ have the same spectrum.\footnote{This is because $\rho$ and $\rho'$ are similar matrices.}

Furthermore,

$$S \left( \sum_{i=1}^{n} \langle v_i|\rho|v_i \rangle |v_i\rangle \langle v_i| \right) = S \left( U \left( \sum_{i=1}^{n} \langle e_i|\rho'|e_i \rangle |e_i\rangle \langle e_i| \right) U^\dagger \right) = H \left( \lambda \left( U \left( \sum_{i=1}^{n} \langle e_i|\rho'|e_i \rangle |e_i\rangle \langle e_i| \right) U^\dagger \right) \right) = H \left( \lambda \left( \sum_{i=1}^{n} \langle e_i|\rho'|e_i \rangle |e_i\rangle \langle e_i| \right) \right) = H(d(\rho')).$$

(18)

Since $\rho'$ is a density matrix, it is in particular hermitian; therefore, $d(\rho') \prec \lambda(\rho')$ by theorem 4.2. Since entropy function $H$ is Schur-concave, we have that $H(d(\rho')) \geq H(\lambda(\rho'))$. This inequality and equations (17) and (18) readily imply inequality (15).
5 Perturbation

5.1 Eigenvectors of perturbed good Hamiltonian

Suppose $H_0$ is a good Hamiltonian for composite system $AB$, where $A$ and $B$ have dimensions $n$ and $m$, respectively. Let $\lambda_{ij}$ be the eigenvalue of $H_0$ corresponding to eigenvector $|ij\rangle$. Assume that $\lambda_{ij}$’s are distinct so that for every eigenvalue, the corresponding eigenvector does not couple subsystems $A$ and $B$. We form a new Hamiltonian $H$ by perturbing $H_0$. Thus, $H = H_0 + \delta H_{AB}$ where $H_{AB}$ is some hermitian matrix. Due to perturbation, some of the eigenvectors of the new Hamiltonian are coupled. The goal of this subsection is to investigate how entangled the eigenvectors of $H$ are. More precisely, we evaluate the eigenvalues of one-party reduced density operator of eigenvectors of $H$ up to the order $\delta^2$.

Since eigenvectors of $H_0$ span the space, we have

$$H_{AB} = \sum_{i,j,i',j'} c_{ij}^{i'j'} |ij\rangle \langle i'j'|.$$  

(19)

By non-degenerate perturbation theory, the eigenvectors of $H$ can be written in the form,

$$|\tilde{\psi}_{ij}\rangle = |ij\rangle - \delta \sum_{kl \neq ij} \frac{c_{ij}}{\lambda_{kl} - \lambda_{ij}} |kl\rangle - \delta^2 \sum_{kl \neq ij} X_{kl} |kl\rangle + O(\delta^3),$$

(20)

where we use the bar notation to remind us that the eigenvector $|\tilde{\psi}_{ij}\rangle$ is not normalized and $X_{kl}$’s are constants whose values are insignificant. Here and throughout this section, $kl \neq ij$ is the short hand for $k \neq i$ or $l \neq j$. For convenience of notation, set

$$a_{kl} \equiv \frac{c_{ij}^{kl}}{\lambda_{kl} - \lambda_{ij}}.$$  

(21)

Consequently, the one-party reduced state of the density operator $|\tilde{\psi}_{ij}\rangle\langle\tilde{\psi}_{ij}|$, before normalization, can be written as

$$\bar{\rho}_A = \text{tr}_B (|\tilde{\psi}_{ij}\rangle\langle\tilde{\psi}_{ij}|) = \left( 1 + \delta^2 \sum_{l \neq j} |a_{il}|^2 \right) |i\rangle \langle i| - \delta \left( \sum_{k \neq i} a_{kj} |k\rangle \langle i| \right) - \delta \left( \sum_{k \neq i} a_{kj}^* |i\rangle \langle k| \right) +$$

$$+ \delta^2 \left( \sum_{k \neq i} \left( \sum_{l \neq j} a_{kl} a_{kl}^* \right) |k\rangle \langle k'| \right) + \delta^2 \left( \sum_{k \neq i} Z_k |k\rangle \langle i| \right) + \delta^2 \left( \sum_{k \neq i} Z_k^* |i\rangle \langle k| \right) + O(\delta^3),$$

where

$$Z_k = \sum_{l \neq j} a_{kl} a_{kl}^* - X_{kj}.$$  

8If an eigenvalue of $H_0$ has degeneracy, then there is no guarantee that every eigenvector in the corresponding eigenspace is uncoupled.
Now our goal is to find eigenvalues of $\bar{\rho}_A$ up to order of $\delta^2$. Define an auxiliary matrix

$$M = -\delta^2 \sum_{k \neq i} (a_{k,j}^* - \delta Z_k^*)(a_{kj} - \delta Z_k)|i\rangle \langle i| - \delta \sum_{k \neq i} (a_{k,j}^* - \delta Z_k^*)|i\rangle \langle k| + \delta \sum_{k \neq i} (a_{kj} - \delta Z_k)|k\rangle \langle i| + I,$$

where $I$ is the $n \times n$ identity matrix. One can verify that,

$$M^{-1} = -\delta^2 \sum_{k \neq i} (a_{k,j} - \delta Z_k)(a_{kj}^* - \delta Z_k^*)|k\rangle \langle k'| + \delta \sum_{k \neq i} (a_{k,j}^* - \delta Z_k^*)|i\rangle \langle k| - \delta \sum_{k \neq i} (a_{kj} - \delta Z_k)|k\rangle \langle i| + I.$$

Note that $\bar{\rho}_A$ and $M \bar{\rho}_A M^{-1}$ have the same eigenvalues. After some rather tedious calculations, one can verify that

$$M \bar{\rho}_A M^{-1} = \left(1 + \delta^2 \left(\sum_{l \neq j} |a_{il}|^2 + \sum_{k \neq i} |a_{kj}|^2\right)\right) |i\rangle \langle i| + \delta^2 \sum_{k \neq i} \left(\sum_{l \neq j} a_{kl} a_{kl}^*\right) |k\rangle \langle k'| + O(\delta^3).$$

By perturbation theory, we know that up to second order in $\delta$, $\bar{\rho}_A$ (as well as $M \bar{\rho}_A M^{-1}$) and

$$\left(1 + \delta^2 \left(\sum_{l \neq j} |a_{il}|^2 + \sum_{k \neq i} |a_{kj}|^2\right)\right) |i\rangle \langle i| + \delta^2 \sum_{k \neq i} \left(\sum_{l \neq j} a_{kl} a_{kl}^*\right) |k\rangle \langle k'|$$

have the same eigenvalues. Clearly, one of the eigenvalues of matrix (22) is

$$1 + \delta^2 \left(\sum_{l \neq j} |a_{il}|^2 + \sum_{k \neq i} |a_{kj}|^2\right),$$

and the other eigenvalues all have factor $\delta^2$. We can express these $n - 1$ remaining eigenvalues in a more enlighten way as follow. Let $U$ be an $n \times m$ matrix whose $kl$-th entry is $\delta a_{kl}$. Produce $n - 1 \times m - 1$ matrix $\tilde{V}$ by deleting row $i$ and column $j$ of matrix $U$. Observe that

$$\tilde{V} \tilde{V}^\dagger = \delta^2 \sum_{k \neq i} \left(\sum_{l \neq j} a_{kl} a_{kl}^*\right) |e_k\rangle \langle e_{k'}|.$$  

Hence, those eigenvalues of matrix (22) with factor $\delta^2$ are equal to eigenvalues of $\tilde{V} \tilde{V}^\dagger$.

Now recall that $\bar{\rho}_A$ was not normalized; therefore, the above eigenvalues are off by a normalization factor from the eigenvalues of one-party reduced state of normalized perturbed eigenvectors. In order to take into account the normalization factor, we need to divide the above eigenvalues by

$$\text{tr}(\bar{\rho}_A) = 1 + \delta^2 \sum_{kl \neq ij} |a_{kl}|^2.$$
Thus the first eigenvalue of $\rho_A$ is,

$$
\begin{align*}
(1 + \delta^2 \sum_{kl \neq ij} |a_{kl}|^2)^{-1} \left(1 + \delta^2 \left(\sum_{l \neq j} |a_{il}|^2 + \sum_{k \neq i} |a_{kj}|^2\right)\right) \\
\left(1 + \delta^2 \left(\sum_{l \neq j} |a_{il}|^2 + \sum_{k \neq i} |a_{kj}|^2\right)\right)^{-1}
\end{align*}
$$

(24)

where the equality holds up to order $\delta^2$. Dividing the remaining eigenvalues by $1 + \delta^2 \sum_{kl \neq ij} |a_{kl}|^2$ does not change their value up to order $\delta^2$, because they are multiples of $\delta^2$.

Hence, the eigenvalues of one-party reduced state of eigenvectors of $H$ (up to order $\delta^2$) are given by $24$ and eigenvalues of $23$.

5.2 Time evolution for perturbed good Hamiltonians

In this subsection, we start with state $|ij\rangle$ and investigate the time evolution of its reduced spectrum subject to Hamiltonian $H$ introduced in subsection 5.1.

From the previous section, we conclude that the normalized eigenvectors of $H$ are

$$
|\psi_{ij}\rangle = |ij\rangle - \delta \sum_{kl \neq ij} a_{kl} |kl\rangle + O(\delta^2).
$$

(25)

Also, using first order perturbation theory, one can verify that the eigenvalues of $H$ are

$$
\lambda'_{ij} = \lambda_{ij} + \delta c_{ij} + O(\delta^2).
$$

(26)

Because $|\psi_{rs}\rangle$ are the eigenvectors of the Hamiltonian $H$, at time $t$

$$
|\psi_{rs}\rangle(t) = t_{rs} |\psi_{rs}\rangle,
$$

where we are using the shorthand notation $t_{rs} = \exp(-i\lambda'_{rs} t/\hbar)$. Hence

$$
\begin{align*}
|i\rangle(t) &= t_{ij} |\psi_{ij}\rangle + \delta \sum_{kl \neq ij} a_{kl} t_{kl} |\psi_{kl}\rangle + O(\delta^2) \\
&= t_{ij} |i\rangle + \delta \sum_{kl \neq ij} a_{kl} |kl\rangle + \delta \sum_{kl \neq ij} a_{kl} t_{kl} |kl\rangle + O(\delta^2) \\
&= t_{ij} |i\rangle - \delta \sum_{kl \neq ij} a_{kl} (t_{ij} - t_{kl}) |kl\rangle + O(\delta^2)
\end{align*}
$$

where we used (26) for the second equality. Recall that in the previous subsection, we were able to evaluate the spectrum of one-party reduced density matrix of $|\bar{\psi}_{ij}\rangle \langle \bar{\psi}_{ij}|$ up to order $\delta^2$ where $\bar{\psi}_{ij}$ was given by equation (20). We can divide $|i\rangle(t)$ by $t_{ij}$ as this does not
change the density matrix of the state vector $|ij\rangle(t)$. Thus, we want to find the spectrum of one-party reduced density matrix of the following state vector up to order of $\delta^2$,

$$
|ij\rangle - \delta \sum_{kl\neq ij} a_{kl} (1 - t_{kl}/t_{ij}) |kl\rangle + O(\delta^2)
$$

$$
= |ij\rangle - \delta \sum_{kl\neq ij} a_{kl} (1 - e^{i(\lambda_{ij} - \lambda_{kl})t/\hbar}) |kl\rangle + O(\delta^2)
$$

$$
= |ij\rangle - \delta \sum_{kl\neq ij} a_{kl} (1 - e^{i(\lambda_{ij} - \lambda_{kl})t/\hbar}) |kl\rangle + O(\delta^2)
$$

where we used (25) for the last equality. Using the result of subsection 5.1, one of the eigenvalues is

$$
1 - \delta^2 \sum_{k\neq i, l\neq j} |a_{kl}|^2 2 (1 - \cos((\lambda_{ij} - \lambda_{kl})t/\hbar)),
$$

and the other eigenvalues can be expressed in the following way. Let $U$ be an $n \times m$ matrix whose $kl$-th entry is $\delta a_{kl}(1 - e^{i(\lambda_{ij} - \lambda_{kl})t/\hbar})$. Produce $n - 1 \times m - 1$ matrix $V$ by deleting row $i$ and column $j$ of matrix $U$. The remaining eigenvalues are equal, up to order $\delta^2$, to the eigenvalues of $VV^\dagger$.

Although the above derivation is valid for every time, we consider small passage of time (for which we can approximate $1 - \exp(i(\lambda_{ij} - \lambda_{kl})t/\hbar)$ by $i(\lambda_{kl} - \lambda_{ij})t/\hbar$) as it yields a very simplified result. Observe that by (21) we have $a_{kl}(\lambda_{kl} - \lambda_{ij}) = c_{kl}^{ij}$. Hence, one of the eigenvalues is

$$
1 - \delta^2 \sum_{k\neq i, l\neq j} \frac{t^2}{\hbar^2} |c_{kl}^{ij}|^2,
$$

and the other eigenvalues can be expressed in the following way. Let $U$ be an $n \times m$ matrix whose $kl$-th entry is $\delta ic_{kl}^{ij}$. Produce $n - 1 \times m - 1$ matrix $V$ by deleting row $i$ and column $j$ of matrix $U$. The remaining eigenvalues are equal, up to order $\delta^2$, to the eigenvalues of $VV^\dagger$. In particular, the eigenvalues are changing as a quadratic function of time.

### 5.3 Example

In this subsection, we apply the theory derived in the last two subsections in the context of an example.

Let $AB$ be a composite system, where $A$ and $B$ consists of $n$ and $m$ spins, respectively. Consequently, subsystems $A$ and $B$ have dimensions $2^n$ and $2^m$, respectively. Consider the following Hamiltonian

$$
H_0 = B \sum_{r=1}^{n+m} \sigma_z^r.
$$

It is easy to verify that $H_0$ is a good Hamiltonian with eigenvectors $|ij\rangle$ where $0 \leq i \leq 2^n - 1$ and $0 \leq j \leq 2^m - 1$ ($i$ and $j$ should be written in binary notation to correspond to the configuration of the spins, see section 2). The corresponding eigenvalue for the eigenvector $|ij\rangle$ is

$$
B \times (\# \text{ of } 1\text{'s} - \# \text{ of } 0\text{'s}),
$$

when $ij$ is written in binary notation.
Consider the perturbation Hamiltonian to be
\[ H_{AB}(d) = \sum_{r=0}^{n+m-d} \sum_{\sigma^r_+ \cdots \sigma^r_-} n + m - d, \]
where the second sum is taken over all possible choices for + and −. For example, for \( n = 2 \) and \( m = 1 \),
\[ H_{AB}(2) = \sigma^1_+ \sigma^1_- + \sigma^1_+ \sigma^2_- + \sigma^1_- \sigma^2_+ + \sigma^2_+ \sigma^3_+ + \sigma^2_- \sigma^3_- + \sigma^3_+ \sigma^3_- . \]

In this example, we are interested to see how the reduced eigenvalues of ground state of \( H_0 \) will change (up to order of \( \delta^2 \)) subject to perturbation \( \delta H_{AB}(d) \). Note that the ground state of \( H_0 \) is \(|0\cdots0\rangle_n \rangle_m \). Since the ground state is non-degenerate, we can apply the result of subsection 5.2. Observe that all states \(|kl\rangle \) with \(|k\rangle \neq |0\cdots0\rangle_n \rangle_m \) and \(|j\rangle \neq |0\cdots0\rangle_n \rangle_m \) for which
\[ \langle 0\cdots0|H_{AB}(d)|kl\rangle \neq 0 \]
are
\[ |k_1l_1\rangle = |0\cdots01\cdots110\cdots0\rangle \]
\[ |k_2l_2\rangle = |0\cdots0n\cdots1110\cdots0\rangle \]
\[ \vdots \]
\[ |k_{d-1}l_{d-1}\rangle = |0\cdots01\cdots110\cdots0\rangle, \]
where in each of the above states we have exactly \( d \) 1’s. Inserting (27) and \( c_{i j}^{l^l} = \langle ij|H_{AB}|l^l j^l\rangle \) (this comes from (19)) into equation (21), yields that
\[ a_{k,l} = \frac{2^d}{2dB}, \]
for \( r = 1, \ldots, d - 1 \).

Next, we form matrix \( \bar{V} \). Observe that the \( k_r l_r \)-th entry of \( \bar{V} \) is
\[ \delta a_{k_r,l_r} (1 - e^{-i2dBt/h}), \]
for \( r = 1, \ldots, d - 1 \) and the remaining entries are zero. Note that since the nonzero entries of \( \bar{V} \) do not share a same column or row, \( \bar{V}\bar{V}^\dagger \) is a diagonal matrix that has exactly \( d - 1 \) nonzero diagonal entries with value
\[ \delta a_{k_r,l_r} a_{k_r,l_r}^* (1 - e^{-i2dBt/h})(1 - e^{i2dBt/h}) = \delta^2 \frac{4^d}{d^2 B^2} \sin^2(dBt/h). \]
Consequently, at time \( t \) after perturbation the ground state has one eigenvalue with value,
\[ 1 - \delta^2(d - 1) \frac{4^d}{d^2 B^2} \sin^2(dBt/h), \]
$d - 1$ eigenvalues with value
\[ \delta^2 \frac{4^d}{d^2 B^2} \sin^2(dBt/h), \]
and the remaining eigenvalues are zero. Note that the above eigenvalues are correct up to order of $\delta^2$.

6 Spread of information

In this section we investigate how the information spreads in a system of $n$ spins subject to a specific Hamiltonian. Define Hamiltonian $\mathcal{H}$ by

\[
\mathcal{H} \equiv \hbar \sum_{r=1}^{n} \sigma_r^+ \sigma_{r+1}^- + \sigma_r^- \sigma_{r+1}^+.
\]

In the above summation and throughout this section, index $n + 1$ is replaced by index 1.

Suppose we start with the state
\[
|\psi(0)\rangle = \frac{1 \cdots 1 + 10 \cdots 0}{\sqrt{2}}.
\] (28)

Observe that at time $t = 0$, the minimum eigenvalue of one-spin reduced density matrix of all spins is $1/2$ except for the first spin which is 0. Since Hamiltonian $\mathcal{H}$ entangles the consecutive spins, the minimum eigenvalue of one-spin reduced density matrix of the spins changes by time. Using these eigenvalues as a measure, we investigate how the state of the first spin influences the state of other spins by time; that is, how the information due to the first spin is spread to other spins.

Let $|\varphi_i\rangle$ denote the state vector where all the spins are 0 except the $i$-th spin which is 1. Observe that
\[
\mathcal{H}|\varphi_i\rangle = \hbar|\varphi_{i-1}\rangle + \hbar|\varphi_{i+1}\rangle.
\] (29)

The above expression implies that the restriction of Hamiltonian $\mathcal{H}$ to the subspace spanned by $\{|\varphi_1\rangle, \ldots, |\varphi_n\rangle\}$ (when written in basis $|\varphi_1\rangle, \ldots, |\varphi_n\rangle$) is equal to
\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 1 \\
1 & 0 & 1 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & \ddots & \ddots & \ddots & \ddots & 0 \\
1 & 0 & \cdots & 0 & 1 & 0 \\
1 & 0 & \cdots & 0 & 1 & 0 \\
\end{bmatrix}
\] (30)

\footnote{Consequently, the system can be thought as a set of $n$ spins placed on a circle.}
It is easy to verify that the eigenvalues and corresponding eigenvectors of the above matrix are, respectively,

\[ 2\hbar \cos \left( \frac{2k\pi}{n} \right) \quad \text{and} \quad \frac{1}{\sqrt{n}} \begin{bmatrix} e^{i(2k\pi/n)} \\ e^{i2(2k\pi/n)} \\ \vdots \\ e^{in(2k\pi/n)} \end{bmatrix} \quad k = 1, \ldots, n. \]

Note that the above eigenvectors are written in the basis \(|\varphi_1\rangle, \ldots, |\varphi_n\rangle\). Thus, \(n\) of the \(2^n\) eigenvectors of \(\mathcal{H}\) are

\[ |v_k\rangle = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} e^{ij(2k\pi/n)} |\varphi_j\rangle, \quad (31) \]

for \(k = 1, \ldots, n\) \(^{10}\). We can write the above expressions in the following matrix relationship,

\[ \begin{bmatrix} |\varphi_1\rangle \cdots |\varphi_n\rangle \end{bmatrix} A = \begin{bmatrix} |v_1\rangle \cdots |v_n\rangle \end{bmatrix}, \quad (32) \]

where \(A\) is an \(n \times n\) matrix whose \(jk\)-th entry is \(e^{i(j2k\pi/n)/\sqrt{n}}\). It is easy to verify that \(A\) is unitary. Thus, multiplying both sides of (32) by \(A^\dagger\) yields,

\[ \begin{bmatrix} |\varphi_1\rangle \cdots |\varphi_n\rangle \end{bmatrix} = \begin{bmatrix} |v_1\rangle \cdots |v_n\rangle \end{bmatrix} A^\dagger. \]

Hence,

\[ |\varphi_j\rangle = \sum_{k=1}^{n} (A^\dagger)^{k_j} |v_k\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e^{-i2jk\pi/n} |v_k\rangle. \]

In particular,

\[ |\varphi_1\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e^{-i2k\pi/n} |v_k\rangle. \]

Consequently, if we start with \(|\varphi_1\rangle\) at time 0 and allow it to evolve under Hamiltonian \(\mathcal{H}\), then at time \(t\),

\[ |\varphi_1(t)\rangle = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} e^{-i2k\pi/n} e^{-it\lambda_k} |v_k\rangle, \]

where \(\lambda_k = 2 \cos(2k\pi/n)\). Substituting (31) in above,

\[ |\varphi_1(t)\rangle = \frac{1}{n} \sum_{m,k} e^{i2\pi k(m-1)/n} e^{-it\lambda_k} |\varphi_m\rangle. \]

Set

\[ B_m(t) = \frac{1}{n} \sum_{k=1}^{n} e^{i2\pi k(m-1)/n} e^{-it\lambda_k}. \quad (33) \]

\(^{10}\)Note that \(v_k\)'s are vectors of length \(2^n\).
Thus,
\[ |\varphi_1(t)\rangle = \sum_{m=1}^{n} B_m(t) |\varphi_m\rangle. \]

On the other hand, note that
\[ \mathcal{H} |1 \cdots 1\rangle = 0, \]
thus \( |1 \cdots 1\rangle \) is an eigenvector of the Hamiltonian \( \mathcal{H} \) with eigenvalue 0.

Hence, if we start with initial state (28) and let it evolve subject to Hamiltonian \( \mathcal{H} \), at time \( t \) we have,
\[ |\psi(t)\rangle = |1 \cdots 1\rangle + \sum_{m=1}^{n} B_m(t) |\varphi_m\rangle \sqrt{2}. \quad (34) \]

Let \( \rho_d(t) \) denote the one-spin reduced density matrix of spin \( d \) at time \( t \). Consequently,
\[ \rho_d(t) = \text{tr}_d(|\psi(t)\rangle\langle\psi(t)|), \quad (35) \]
where \( \text{tr}_d \) means tracing out the system of all spins except spin \( d \).

Observe that
\[
\begin{align*}
\text{tr}_d(|1 \cdots 1\rangle\langle\varphi_m|) &= 0 \\
\text{tr}_d(|\varphi_m\rangle\langle\varphi_{m'}|) &= \begin{cases} 
\delta_{mm'} |1\rangle \langle 1| & \text{for } m = d \\
\delta_{mm'} |0\rangle \langle 0| & \text{for } m \neq d \\
\text{tr}_d(|1 \cdots 1\rangle\langle 1 \cdots 1|) &= |1\rangle \langle 1| 
\end{cases} \quad (36)
\end{align*}
\]
Substituting (34) into (35) and using relationships (36), we have
\[ \rho_d(t) = \left( 1 + \frac{|B_d(t)|^2}{2} \right) |1\rangle \langle 1| + \left( \sum_{m \neq d} \frac{|B_m(t)|^2}{2} \right) |0\rangle \langle 0|. \quad (37) \]

As the sum of the eigenvalues of \( \rho_d(t) \) is equal to one, we conclude that the minimum eigenvalue of \( \rho_d(t) \) is equal to
\[ \frac{1 - |B_d(t)|^2}{2}. \quad (38) \]

In Figure 2 and 3, we used equation (38) to plot the minimum eigenvalue of \( \rho_d(t) \) for different \( d \)'s in a system of 200 spins are shown. As one can see in Figure 3, the minimum eigenvalue of \( \rho_{50}(t) \) is almost one-half \( \frac{1}{2} \) until \( t = 45s \) (we call this critical time \( T_{\text{lagging}} \)). For \( t > T_{\text{lagging}} \), the minimum eigenvalue of \( \rho_{50}(t) \) oscillates \( \frac{1}{2} \) with attenuation until \( t = 144s \), after which it behaves disorderly (we call this critical time \( T_{\text{chaos}} \)). The behaviour of the minimum eigenvalue of \( \rho_{50}(t) \) that is described here is typical for other spins as well. Ostensibly, \( T_{\text{lagging}} \) and \( T_{\text{chaos}} \) are functions of \( d \) and \( n \):
\[ T_{\text{lagging}}(d, n) \approx \min(d, n - d) \quad T_{\text{chaos}}(d, n) \approx \max(d, n - d). \]
Figure 2: The minimum eigenvalue of $\rho_{30}(t)$ (black and solid), $\rho_{40}(t)$ (blue and dotted), $\rho_{50}(t)$ (red and dashdot), and $\rho_{30}(t)$ (green and dashed) in a system of 200 spins.

Figure 3: The minimum eigenvalue of $\rho_{50}(t)$ in a system of 200 spins. It can be observed that $T_{\text{lagging}}(50,200) \approx 45s$ and $T_{\text{chaos}}(50,200) \approx 144s$. 
Based on numerical data, we propose the following description of how information due to the first spin spreads to other spins. Recall that the minimum eigenvalue of $\rho_d(t)$ is a measure of how much the state of the first spin influences state of the $d$-th spin. Since the spins are arranged in circle, the information due to the first spin spreads like a wave in both direction. Now $T_{\text{lagging}}$ signifies the amount of time required for the information to arrive at spin $d$ for the first time. However, since information has spread in two directions, at some later time (that is $T_{\text{chaos}}$) a second wave of information arrives at spin $d$. The combination of this wave of information with the first wave of information that had arrived at $T_{\text{lagging}}$ results into the disorderly behaviour observed, for example, in Figure 3 from $t = 144s$ onward.

Note that since Hamiltonian $\mathcal{H}$ couples only the consecutive spins, the spread of information is independent of the number of the spins in the system. Therefore, for $t < T_{\text{chaos}}(d, n)$, the behaviour of function $\rho_d(t)$ is independent of $n$. This can be seen in Figure 4. Let $n \to \infty$ in (33):

$$B_m(t) = \int_0^1 e^{i\pi(m-1)x} e^{-i2\cos(2\pi x)} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos((m-1)x) e^{-i\cos(x)t} dx. \quad (39)$$

Consequently, for $t < T_{\text{chaos}}(d, n)$, we can use (39) instead of (33) in equation (38) to find the minimum eigenvalue of $\rho_d(t)$. As it can be seen in Figure 2, the amount by which the minimum eigenvalue of $\rho_d(t)$ drops around time $T_{\text{lagging}}(d, n)$ is different for different spins. From equation (38), the amplitude (i.e. difference from the level line) of the minimum eigenvalue of $\rho_d(t)$ is equal to $|B_d(t)|^2/2$. Let $h(d)$ denote the maximum amplitude of the minimum eigenvalue of $\rho_d(t)$ for $0 \leq t \leq T_{\text{chaos}}(d, n)$; that is,

$$h(d) = \max_{0 \leq t \leq T_{\text{chaos}}(d, n)} \frac{|B_d(t)|^2}{2}.$$

Ostensibly, the maximum amplitude occurs around $T_{\text{lagging}}(d, n)$.

From equation (37), it is easy to verify that $\rho_d(t) = \rho_{n-d+2}(t)$\textsuperscript{13}. Therefore, it suffices to find $h(d)$ for $2 \leq d \leq n/2$\textsuperscript{14}. To find $h(d)$ we make the following assumption\textsuperscript{14}: let $t_0$ (where $0 \leq t_0 \leq T_{\text{chaos}}(d, n)$) be the time for which $\frac{|B_d(t_0)|^2}{2} = h(d)$, then $\frac{|B_j(t_0)|^2}{2} < h(d)$ for $j \neq d$.

In Figure 5, we have plotted $\log(|B_d(t)|^2/2)$ versus $\log(t)$ for different values of $d$. Earlier, we noted that $h(d)$ is achieved around $T_{\text{lagging}}(d, n)$ and $T_{\text{lagging}}(d, n) \approx d$ for $2 \leq d \leq n/2$. Now incorporating the assumption, we deduce that the line that appears in the top left corner of Figure 5 yields a linear relationship between $\log(h(d))$ and $\log(d)$.

\textsuperscript{11} Although not identically 1/2.

\textsuperscript{12} The oscillatory behaviour is quite nontrivial and requires further investigation to understand its physical significance.

\textsuperscript{13} Intuitively, this is obvious as spins are placed in a circle.

\textsuperscript{14} The case $d = 1$ is of no interest as we already know that $h(1) = 1/2$.

\textsuperscript{15} Although, the numerical data suggest that this assumption is always valid, we have not been able to derive it analytically.
Figure 4: The minimum eigenvalue of $\rho_{50}(t)$ in a system of 200 spins (up figure) and 300 spins (down figure). As one can see, the number of spins in the system does not change the behaviour of $\rho_{50}(t)$ for $t < 144$. 
Figure 5: Logarithm of the amplitude of the minimum eigenvalue of \( \rho_d(t) \) versus the logarithm of time for \( d = 5, 10, \ldots, 100 \).

Fitting the data linearly, we conclude that for \( 2 \leq d \leq n/2 \),

\[ h(d) \approx \frac{d^{-2/3}}{4}. \]

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References


