Inverse problems for the Schrödinger equations with time-dependent electromagnetic potentials and the Aharonov-Bohm effect.

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The Schrodinger equation.

Consider the Schrodinger equation

$$(1) \qquad i\frac{\partial u(x,t)}{\partial t} = Hu,$$

where

$$Hu = \sum_{j=1}^{n} \left(-i\frac{\partial}{\partial x_j} - A_j(x,t) \right)^2 u(x,t)$$

$$+V(x,t)u(x,t),$$

in a domain $D \subset \mathbf{R}^{n+1}$.

Equation (1) describes the nonrelativistic electron in a classical electromagnetic field.

$|u(x,t)|^2$ is the probability density.

The gauge group.

Denote by $G(D)$ the group of all $C^\infty(\overline{D})$ functions $c(x,t)$ such that $|c(x,t)| = 1$ for all $(x,t) \in \overline{D}$. Group $G(D)$ is called the gauge group and the map $v(x,t) = c(x,t)u(x,t)$ is called the gauge transformation.

Note that if $\left(i\frac{\partial}{\partial t} - H\right) u = 0$ then $u' = c^{-1}(x,t)u$ satisfies the equation $\left(i\frac{\partial}{\partial t} - H'\right) u' = 0$ where $i\frac{\partial}{\partial t} - H' = 0$ has the form (1) with $(A(x,t), V(x,t))$ replaced by $(A'(x,t), V'(x,t))$ where

(2)

$$A_j'(x,t) = A_j(x,t) + ic^{-1}(x,t)\frac{\partial c(x,t)}{\partial x_j}, \quad 1 \le j \le n,$$

$$V'(x,t) = V(x,t) - ic^{-1}(x,t)\frac{\partial c(x,t)}{\partial t}.$$

Potentials $(A(x,t), V(x,t))$ and $(A'(x,t), V'(x,t))$ are called gauge equivalent.

The Aharonov-Bohm effect.

If $u' = c^{-1}(x,t)u$ then $|u'(x,t)|^2 = |u(x,t)|^2$, i.e. the probability density does not change under the gauge transformation. The same is true for all observable quantities. Therefore any two potentials (A,V) and (A',V') belonging to the same gauge equivalence class are undistinguishable in any physical experiment, i.e. they have the same physical impact.

Let

$$B = \text{curl } A, \qquad E = -\frac{\partial V}{\partial x} - \frac{\partial A}{\partial t}$$

be the magnetic and electric fields, respectively. It follows from (2) that if (A,V) and (A',V') are gauge equivalent then the corresponding electric and magnetic fields are equal : $B = B', \;\; E = E'$.

In classical physics only electromagnetic fields (E,B) have a physical meaning and electromagnetic potentials (A,V) are mathematical tools only. If D is simply-connected (in this case any element of the gauge group has a form $c(x,t) = e^{i\varphi(x,t)}$, where $\varphi(x,t) \in C^\infty(\overline{D})$) then $E = E', \;\; B = B'$ implies that (A,V) and (A',V') are gauge equivalent.

If $\overline{D}$ is not simply-connected then there exists (A,V) and (A',V') such that $E' = E, \;\; B' = B$, but (A,V) and (A',V') belong to different gauge equivalence classes.

It was shown by Aharonov and Bohm (Phys. Rev. 115, 485-491 (1959)) that such electromagnetic potentials have a different physical impact. It can be detected in a physical experiment.

This phenomenon is called the Aharonov-Bohm effect.

The domain.

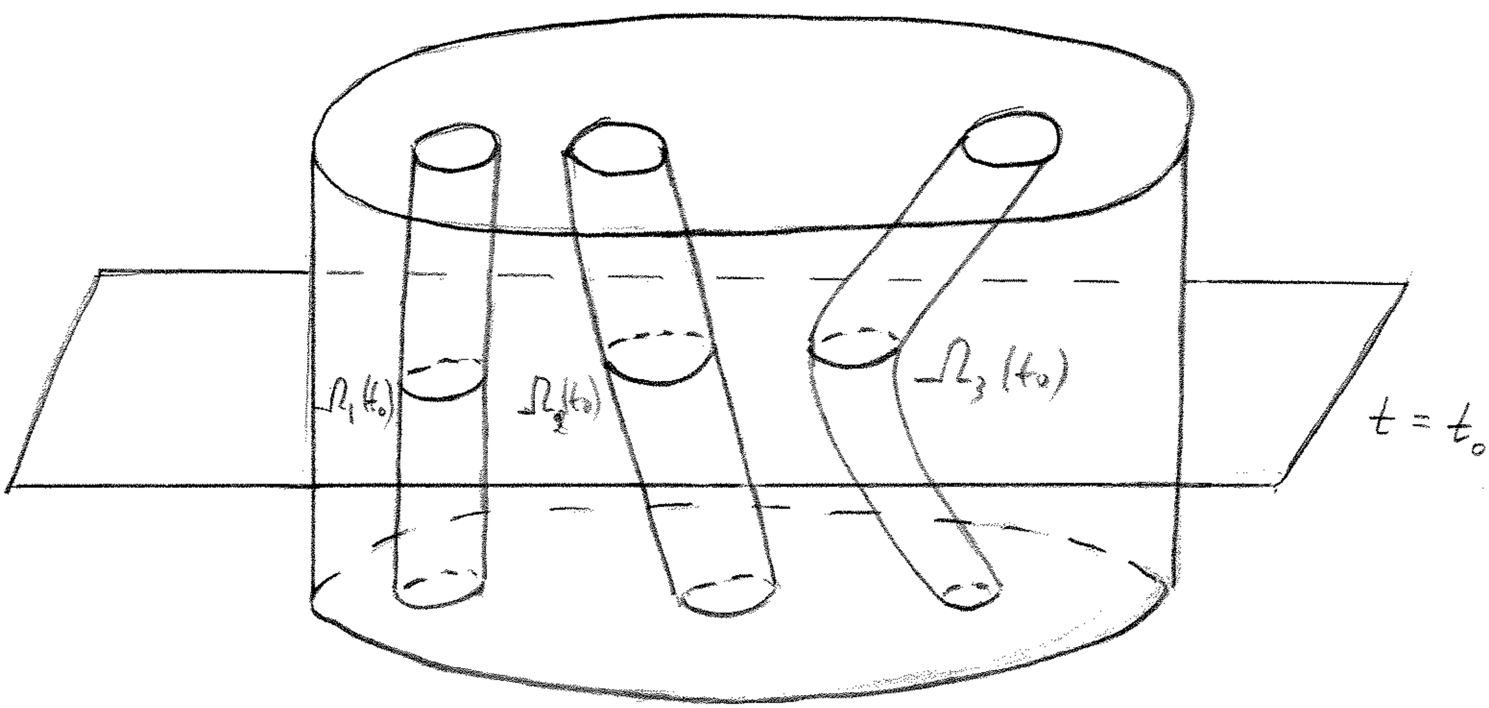
Let Ω_0 be diffeomorphic to a ball in $\mathbf{R}^n$. Let $D \subset \Omega_0 \times [0,T]$ be a domain with the following properties : Denote $D_{t_0} = D \cap \{t = t_0\}$. We assume that $D_{t_0} = \Omega_0 \setminus \cup_{j=1}^m \overline{\Omega}_j(t_0)$ where $\Omega_j(t_0)$ are a piece-wise smooth nonintersecting domains, called obstacles, $1 \leq j \leq m$. We assume that

$$\Omega'(t_0) = \cup_{j=1}^m \Omega_j(t_0)$$

depends smoothly on $t_0 \in [0,T]$.

Finally we assume that the normal to $\partial D \setminus (D_0 \cup D_T)$ in $\mathbf{R}^{n+1}$ is not parallel to the t-axis for any $t \in [0,T]$.

Domain $\mathcal{D}$



$\mathcal{D}_{t_0} = \Omega_0 \setminus (\Omega_1(t_0) \cup \Omega_2(t_0) \cup \Omega_3(t_0))$

Description of gauge equivalent classes.

Let γ be a closed curve in D. Let $\int_\gamma A \cdot dx - Vdt$ be the line integral in D. Denote

(3)

$$R(A, V, \gamma) = \exp\{i \int_\gamma (-A(x,t) \cdot dx + V(x,t)dt)\}.$$

R is called the nonintegrable phase factor (c.f. Wu, T, and Young, C., Phys. Rev. D 12, 38-45 (1975) and Olarin, S. and Popescu, I., Rev. of Modern Phys., vol. 57, 339-436)).

It is easy to see that if (A, V) and (A', V') are gauge equivalent then

$$R(A, V, \gamma) = R(A', V', \gamma) \tag{4}$$

for all closed $\gamma \in \overline{D}$. Vice versa, if (4) holds for all γ, then (A, V) and (A', V') are gauge equivalent in D.

To compute $R(A, V, \gamma)$ one can use the Stoke's formula

$$\int_\gamma A \cdot dx - Vdt = \int_S (B_1 dx_2 dx_3 + B_2 dx_1 dx_3 + B_3 dx_1 dx_2 + E_1 dt dx_1 + E_2 dt dx_2 + E_3 dt dx_3), \tag{5}$$

where $n = 3,\ \gamma = \partial S$.

In the case $n = 2$ we have :

(6)

$$\int_\gamma A_1 dx_1 + A_2 dx_2 - Vdt = \int_S (B_3 dx_1 dx_2 + E_1 dt dx_1 + E_2 dt dx_2).$$

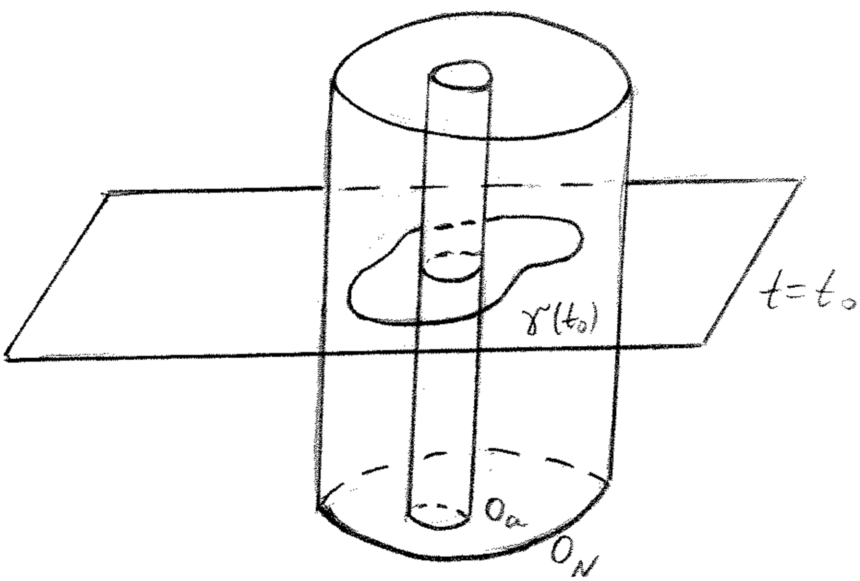
The right hand sides of (5) and (6) are called the electromagnetic fluxes.

Formulas (5), (6) can be used for the computations of $R(A, V, \gamma)$, especially when either the magnetic or electric fields is "shielded" inside the obstacles, i.e. when either B or E is nonzero in $\Omega'(t) = \cup_{j=1}^{m} \Omega_j(t)$ and zero in D.

Two examples.

We consider two examples when the Aharonov-Bohm effect is caused by the magnetic and electric fluxes, respectively.

Denote by O_r the disk $\{x_1, x_2 : x_1^2 + x_2^2 < r^2\}$. Let $\Omega = O_N \backslash O_a$, $D = \Omega \times (0,T)$ and (A, V) and (A', V') be two electromagnetic potentials such that the corresponding magnetic fields $B = \text{curl } A$ and $B' = \text{curl } A'$ are shield in $O_a \times [0,T]$, i.e. $B = B' = 0$ in D.



Suppose $E = E'$ in D. We shall describe the gauge equivalence classes of electromagnetic potentials with given electromagnetic fields $(0, E)$ in D in term of magnetic fluxes $\int_{O_a} B dx_1 dx_2$. Since $B = B' = 0$ in D and $E = E'$

in D, the Stoke's formula gives that

$$\int_\gamma (A - A') \cdot dx - (V - V')dt \tag{7}$$

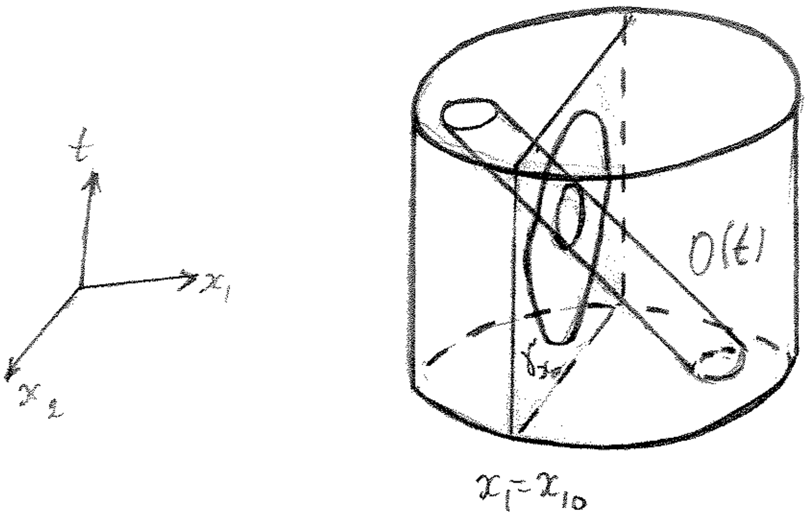
depends only on the homotopy class of the closed curve γ in D.

Note that the gauge group $G(D)$ consists of $c(x,t) = e^{im\theta + i\varphi(x,t)}$ where $m \in \mathbf{Z}$, $\varphi(x,t) \in C^\infty(\overline{D})$ and θ is the angle in the polar coordinates. Fix any $t_0 \in [0,T]$. Then γ is homotopic to $k\gamma(t_0)$, where $\gamma(t_0)$ is a simple closed curve in the plane $t = t_0$ encircling O_a.

Denote
$b(t_0) = \int\int_{O_a} B dx_1 dx_2, \quad b'(t_0) = \int\int_{O_a} B' dx_1 dx_2$
the magnetic fluxes of B and B'. We have that
$\int_{\gamma(t_0)} (A - A') \cdot dx = b(t_0) - b'(t_0)$ is independent of t_0.

Denote $b(t_0) - b'(t_0) = \alpha$. If $\alpha \neq 2\pi p, \ \forall p \in \mathbf{Z}$ then (A, V) and (A', V') are not gauge equivalent, i.e. Aharonov-Bohm effect takes place. If $\alpha = 2\pi k_0$ for some $k_0 \in \mathbf{Z}$ then the integral (7) is equal to $2\pi k_0 k$ and therefore $R(A, V, \gamma) = R(A', V', \gamma)$ for all γ, i.e. (A, V) and (A', V') are gauge equivalent.

This example is a slight generalization of the original Aharonov-Bohm example when $V = E = 0$,
$A(x) = \frac{b_0}{2\pi}(\frac{-x_2}{|x|^2}, \frac{x_1}{|x|^2})$ is independent of t, $B = b_0\delta(x)$,
$b_0 = \int\int_O B dx_1 dx_2$ is the magnetic flux.



Consider now the Aharonov-Bohm effect caused by the electric flux. Consider an obstacle $O(t)$ moving with speed v_0 in the x_1-direction :
$O(t) = \{(x_1, x_2, t) : (x_1 - v_0 t)^2 + x_2^2 < a^2\}$.

Suppose $E = (0, E_2)$, $E' = (0, E_2')$ are electric fields with supports in $O(t)$, i.e. $E = E' = 0$ in
$D = (O_N \times [0, T]) \setminus O(t)$.

Let (A, V), (A', V') be electromagnetic potentials corresponding to electromagnetic fields (E, B), (E', B'), where $B = B'$ in D. Denote by $O_{x_{10}}$ the intersection of $O(t)$ with the plane $x_1 = x_{10}$ and let $\gamma_{x_{10}}$ be a simple curve in this plane containing $O_{x_{10}}$. Denote by $e(x_{10}) = \int_{O_{x_{10}}} E_2 dx_2 dt$, $e(x_{10}) = \int_{O_{x_{10}}} E_2' dx_2 dt$ the electric fluxes.

Since $E = E' = 0$ in D and $B = B'$ in D we have that the integral (7) depends only on the homotopy class of γ. As above this leads to the conclusion that $e'(x_{10}) = e(x_{10}) = \alpha$, where α is independent of x_{10}. Potentials (A, V) and (A', V') are gauge equivalent iff $\alpha = 2\pi k_0$ for some $k_0 \in \mathbf{Z}$.

Gauge invariant boundary data.

Consider the Schrödinger equation (1) in D.

$$i\frac{\partial u}{\partial t} - Hu = 0, \quad (x,t) \in D,$$
$$u(x,0) = 0, \quad x \in D_0 = D \cap \{t = 0\},$$
$$u|_{\partial\Omega'(t)} = 0 \quad \forall t \in [0,T],$$

where $\Omega'(t) = \cup_{j=1}^{m} \Omega_j(t)$.

On the boundary $\partial\Omega_0 \times (0,T)$ we impose the following boundary conditions:

$$|u(x,t)|^2|_{\partial\Omega_0\times(0,T)} = f_1(x,t), \tag{8}$$
$$\frac{\partial}{\partial\nu}|u(x,t)|^2|_{\partial\Omega_0\times(0,T)} = f_2(x,t)$$
$$S(x,t)u|_{\partial\Omega_0\times(0,T)} = f_3(x,t)$$

where $Su = \Im\left(\frac{\partial u}{\partial x} - iAu\right)\overline{u}$ is the probability current, $\frac{\partial}{\partial\nu}$ is the normal derivative at $\partial\Omega_0$. It is easy to see thatSu is gauge invariant.

Denote by Λ the Dirichlet-to-Neumann operator on $\partial\Omega_0 \times (0,T)$:

$$\Lambda f = \frac{\partial u}{\partial\nu} + i(A\cdot\nu)u|_{\partial\Omega_0\times(0,T)}, \tag{9}$$

where $i\frac{\partial u}{\partial t} - Hu = 0$ in D, $f = u|_{\partial\Omega_0\times(0,T)}$.

We say that the D-to-N operators Λ and Λ' corresponding to the Schródinger operators $i\frac{\partial u}{\partial t} - Hu = 0$

and $i\frac{\partial u'}{\partial t} - H'u' = 0$, respectively, are gauge equivalent if there exists $c \in G(\overline{D})$ such that

$$\Lambda' = c_0^{-1}\Lambda_0 c_0 \quad \text{on} \quad \partial\Omega_0 \times (0,T),$$

where c_0 is the restriction of $c(x,t)$ to $\partial\Omega_0 \times (0,T)$.

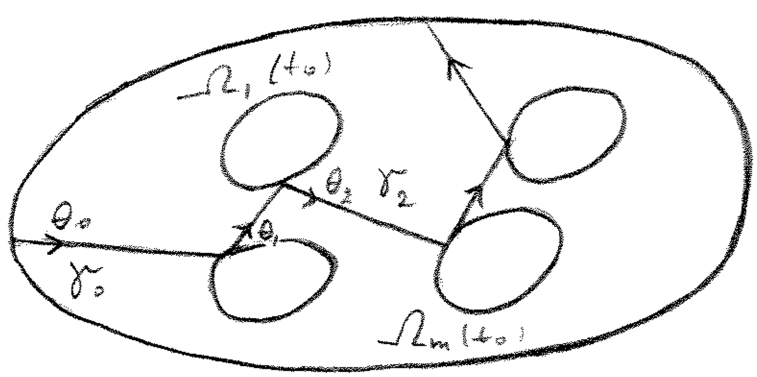
Lemma .1. *Let $(i\frac{\partial}{\partial t} - H)u = 0$ and $(i\frac{\partial}{\partial t} - H')u' = 0$ be two Schrödinger equations in D, $u(x,0) = u'(x,0) = 0$, $u(x,t)|_{\partial\Omega'(t)} = u'|_{\partial\Omega'(t)} = 0$ for all $t \in [0,T]$. Then the D-to-N operators Λ and Λ' are gauge equivalent on $\partial\Omega_0 \times (0,T)$ if and only if for any solution u there exists u' such that the gauge invariant boundary data (8) of $u(x,t)$ and $u'(x,t)$ are equal.*

Inverse problem.

We consider the inverse problem of the determination of the gauge equivalent class of electromagnetic potentials by the gauge equivalent boundary data.

An affiramtive solution of the inverse problem assures that measuring the gauge invariant boundary data (8) we can detect the Aharonov-Bohm effect.

Theorem 1. *Assume that $n = 2$ and that for each $t_0 \in [0, T]$ obstacles $\Omega_j(t_0)$, $1 \le j \le m$, are piece-wise smooth and convex. Suppose that there is no trapped broken rays in $\Omega_0 \setminus \Omega'(t_0)$ for each $t_0 \in [0, T]$. If the D-to-N operators Λ and Λ' are gauge equivalent on $\partial\Omega_0 \times (0, T)$ then the electromagnetic potentials (A, V) and (A', V') are gauge equivalent in D.*



Broken rays

Broken ray $\gamma(t_0)$ has finite number of nontangential reflections $\gamma = \gamma_0 \cup \gamma_1 \cup \ldots \cup \gamma_r$.

Proof of Theorem 1 (see ArXiv:math.AP/0611342)

Making a gauge transformation one can assume that $\Lambda = \Lambda'$ on $\partial\Omega_0 \times (0,T)$.

Lemma 0.2. *Suppose $\Lambda = \Lambda'$ on $\partial\Omega_0 \times (0,T)$. Then for each broken ray $\gamma(t_0)$, $\forall t_0 \in [0,T]$ we have*

$$\exp\left\{i\int_{\gamma(t_0)} A \cdot dx\right\} = \exp\left\{i\int_{\gamma(t_0)} A' \cdot dx'\right\}$$

or more explicitely

(10)

$$\exp\left\{i\sum_{j=0}^{r}\int_{\gamma_j}(A(x_0^{(j)} + s\theta_j, t_0) - A'(x_0^{(j)} + s\theta_j, t_0)) \cdot \theta_j ds\right\} = 1,$$

where θ_j is the direction of γ_j, $x_0^{(j)}$ is the starting point of γ_j, $0 \le j \le r$.

Proof of Lemma 2 For each broken ray $\gamma(t_0)$ we can construct a geometric optic solution of $i\frac{\partial u}{\partial t} - Hu = 0$ of the form

(11)

$$u_N = \sum_{j=0}^{r}\sum_{p=0}^{N}\frac{a_{pj}(x,t,\theta_0)}{(ik)^p}e^{-ik^2t+ik\psi_j(x,t,\theta_0)} + O\left(\frac{1}{k^{N+1}}\right),$$

where $\psi_0(x,t,\theta_0) = x\cdot\theta_0$, k is a large parameter, $u_N|_{\partial\Omega'(t_0)} = 0$, $u_N(x,0) = 0$, the principal part of u_N is zero outside of a small neighborhood of $\gamma = \gamma_0 \cup \gamma_1 \cup \ldots \cup \gamma_r$. Using the Green's formula and that $\Lambda = \Lambda'$ we can get after some computations the relation (10).

Analogously one can prove the following lemma:

Lemma 0.3. *Let $\Lambda = \Lambda'$ and $A = A'$ then for any broken ray we have:*

$$\sum_{j=0}^{r} \int_{\gamma_j(t_0)} (V - V')ds = 0 \tag{12}$$

The tomography of broken rays.

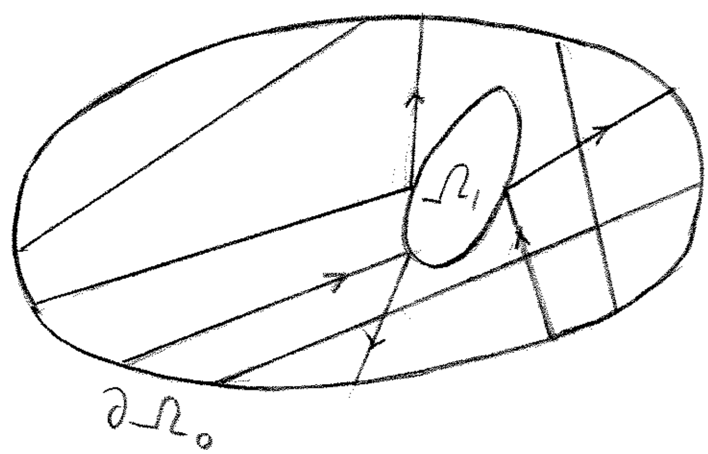
Lemmas 2 and 3 reduce the proof of Theorem 1 to the study of a new branch of X-rays transform (X-ray tomography) the tomography of broken rays:

Given the value of $\int_{\gamma(t_0)} f ds$ for each broken ray in D_{t_0} recover $f(x, t_0)$ in D_{t_0} or at least prove the uniqueness:

If $\int_{\gamma(t_0)} f ds = 0$ for all broken rays $\gamma(t_0)$ then $f(x, t_0) = 0$ in D_{t_0}.

The study of broken rays transforms can be useful in some problems.

Consider the case of one convex obstacle :



Let $\Omega = \Omega_0 \setminus \Omega_1$ where $\Omega_0 \subset \mathbf{R}^2$ is a disk and Ω_1 is a convex domain. Let $f(x)$ be a smooth function in $\overline{\Omega}_0 \setminus \Omega_1$. The following uniqueness theorem is known (Helgason and others):

Theorem 2 ("hole" theorem). *If $\int_\gamma f ds = 0$ for all lines $\gamma \in \Omega_0$ nonintersecting Ω_1 then $f(x) = 0$.*

The recovery of $f(x)$ from the integrals $\int_\gamma f ds$ over these lines is severly ill-posed problem. The problem of the recovery of $f(x)$ from integral over all broken rays in Ω is a well-posed problem. The following estimate holds:

For each $x \in \overline{\Omega}_0 \setminus \Omega_1$ and $\theta \in S^1$ denote by $\gamma_{x,\theta}$ a broken ray that starts on $\partial\Omega_0$, ends at x and has the direction θ at x. Let $g(x,\theta) = \int_{\gamma x,\theta} f ds$. Suppose we know $g(x,\theta)$ when $x \in \partial\Omega_0$ and any $\theta \in S^1$. The following stability estimate holds:

$$(13) \qquad \int_{\Omega_0\setminus\Omega_1} |f(x)|^2 dx$$

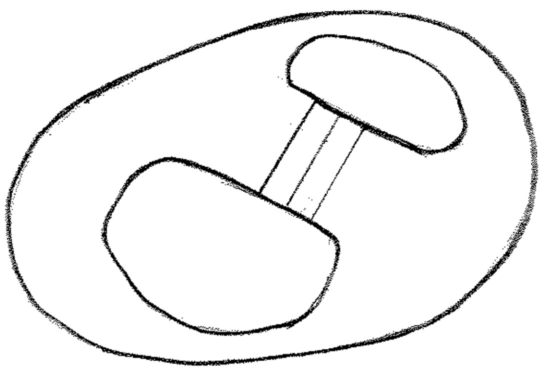
$$\leq C \int_0^L \int_0^{2\pi} \left(\left| \frac{\partial g(x(s),\theta(\varphi))}{\partial s} \right|^2 + \left| \frac{\partial g(x(s),\theta(\varphi))}{\partial \varphi} \right|^2 \right) ds d\varphi,$$

where $\theta(\varphi) = (\cos\varphi, \sin\varphi)$, $x = x(s)$ is the parametric equation of $\partial\Omega_0$, L is the length of $\partial\Omega_0$. Estimate (13) can lead to a proof of convergence of numerical methods for computing $f(x)$ from $g(x(s),\theta(\varphi))$.

Estimate of the form (13) for domains without obstacles was established first by Mukhometov (1977).

The case of several obstacles.

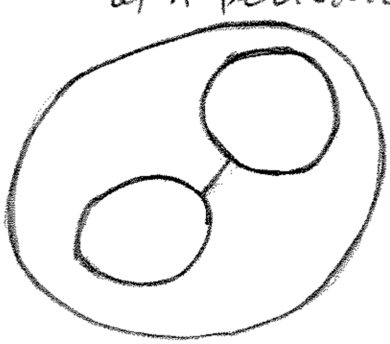
Consider the case of more than one obstacle. We want to prove the uniqueness theorem. The problem that one has to overcome here is the presence of trapped rays. For example, if the obstacles have two parallel segments, then there are many trapped rays, and one can show that in this case there is no uniqueness for the tomography problem.



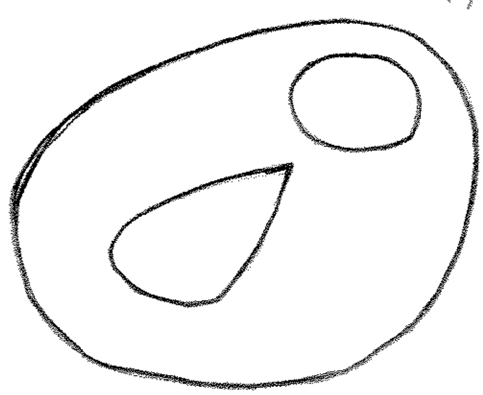
Many periodic orbits
No uniqueness.

Note that trapped rays always exists if there is more than one smooth obstacle.

a) A periodic orbit



b) No trapped ray



In the last example introduce a corner. Then there is no trapped rays for such domain. We can prove the following lemma:

Lemma .4. *Let Ω_j, $1 \le j \le m$, be piece-wise smooth convex obstacles such that there is no trapped rays in the domain $\Omega = \Omega_0 \setminus \cup_{j=1}^m \Omega_j \subset \mathbf{R}^2$. Then $\int_\gamma f ds = 0$ for all broken rays γ in Ω implies that $f = 0$.*

Lemma 5. *Let Ω be the same as in Lemma 3. Suppose $\exp(i \int_\gamma A \cdot dx) = \exp(i \int_\gamma A' \cdot dx)$ for all broken rays in $\Omega \subset \mathbf{R}^2$. Then there exists $c(x) \in C^\infty(\overline{\Omega})$, $|c(x)| = 1,\ c(x) = 1$ on $\partial\Omega_0$, such that*

$$A - A' = ic^{-1}\frac{\partial c}{\partial x}.$$

Combining Lemmas 1, 2, 3, 4, 5 we prove Theorem 1.

Some references.

The inverse scattering problem for the Schrödinger equation for $n = 2$ with one smooth convex obstacle and time-dependent potentials was considered by Nicoleau (2000) and R.Weder (2002) in connection with the Aharonov-Bohm effect (see also Yafaev).

Inverse scattering problem for the Schrödinger equation with time-dependent periodic in t potential was considered by R.Weder (2003).

The Schrödinger equation with time-dependent Yang-Mills potentials.

The equation has the form:

$$Lu \stackrel{def}{=} i\frac{\partial u(x,t)}{\partial t} - \sum_{j=1}^{n}\left(-iI_m\frac{\partial}{\partial x_j} - A_j(x,t)\right)^2 u(x,t) + V(x,t)u(x,t), \tag{14}$$

where $A_j,\ 1 \leq j \leq n,\ V(x,t),\ u(x,t)$ are $m \times m$ matrices, I_m is the identity matrix in $\mathbf{C}^m$. We assume that A_j, V are self-adjoint. The electromagnetic potentials are the particular case when $m = 1$. The equation (15) is considered in the domain $\Omega_0 \times (0,T)$ without obstacles. The gauge group G consists in this case of $m \times m$ unitary matrices smooth in $\overline{\Omega}_0 \times [0,T]$.

Yang-Mills potentials (A,V) and (A',V') are gauge equivalent if there exists $c(x,t) \in G$ such that

$$A_j' = c^{-1}A_jc + ic^{-1}\frac{\partial c}{\partial x_j}, \quad 1 \leq j \leq n,$$
$$V' = c^{-1}Vc - ic^{-1}\frac{\partial c}{\partial t}.$$

As in the case of electromagnetic potentials one can prove that if the D-to-N operators Λ and Λ' are gauge equivalent on $\partial\Omega_0 \times (0,T)$ then (A,V) and (A',V') are gauge equivalent in $\overline{\Omega}_0 \times [0,T]$.

The reduction to the tomography problem (analog of Lemmas 2 and 3) is similar to the case of electromagnetic potentials.

The tomography problem in this case has the following form (when $n = 2$):

Let $b(x, t, \theta)$ be the solution of the following Cauchy problem:

$$\theta \cdot \frac{\partial b(x,t,\theta)}{\partial x} - i(A(x,t) \cdot \theta) b(x,t,\theta) = 0,$$

$x \in \mathbf{R}^2$, $\theta \in S^1$, t is a parameter, $b(x + s\theta, t, \theta) \longrightarrow I_m$ when $s \longrightarrow -\infty$.

Matrix $b_\infty = \lim_{s \longrightarrow +\infty} b(x + s\theta, t, \theta)$ is called the non-Abelian Radon transform of A. The problem of uniqueness of the non-Abelian Radon transform is much harder then in the case $m = 1$ (c.f. works of R.Novikov, G.Eskin and J.Ralston, G.Eskin). It can be proven that if $A(x,t) = (A_1(x,t), A_2(x,t))$ has a compact support then b_∞ uniquely determines $A(x,t)$ for each t (see [E], Russian Journ. of Math. Phys., 2004). The case $n > 2$ follows easily from the case $n = 2$.

Hyperbolic inverse problems approach.

In the case when electromagnetic potentials are time-independent there is a direct connection between the initial-boundary problems for the Schrödinger equation and the hyperbolic equation:

$$\frac{\partial^2 v}{\partial t^2} + H(x, -i\frac{\partial}{\partial x})v = 0. \tag{15}$$

The connection is given by the formula (Y.Kanai, 1977):

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{4t}} v(x,s)ds,$$

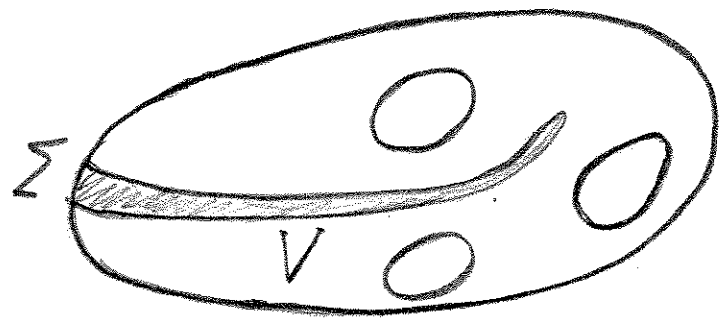
where

$$i\frac{\partial u}{\partial t} - H(x, -i\frac{\partial}{\partial x})u = 0.$$

Also both (1), (15) can be reduced to the stationary Schrödinger equation by the Fourier-Laplace transform in t.

For the hyperbolic inverse problem with time-independent coefficients there is a powerful Boundary Control (BC) method discovered by M.Belishev and extended by Belishev, Y.Kurylev, M.Lassas and others including myself. It follows from these results that $\Lambda = \Lambda'$ on $\partial\Omega_0 \times (0, +\infty)$ implies that (A, V) and (A', V') are gauge equivalent for any smooth multi-connected domain $\Omega \subset \mathbf{R}^n$, $n \geq$

2. No convexity restriction and no the condition of absence of the trapped rays (assuming that A, V are real-valued).



This is a corollary from [G.Eskin, New approach to the hyperbolic inverse problems II (Global case), ArXiv:math.AP/07013] that shows the flexibility and strength of the hyperbolic approach:

Let $x_0 \in \partial\Omega_0$ and x_1 be any point in Ω. Let γ be any continuous curve connecting x_0 and x_1. Denote by V a small neighborhood of γ. Let $\Sigma = \partial\Omega_0 \cup \overline{V}$. Then knowing the D-to-N operator in $\Sigma' \times [0, 2L]$ where Σ' is any neighborhood of Σ and $L > |\gamma|$, $|\gamma|$ is the length of γ, we can recover (A, V) modulo gauge transformation in V. The proof uses the hyperbolic operator only in a small neighborhood V' of $\overline{V}$. For example, A, V can be real-valued only in V'.

The strength of the hyperbolic approach is due to the finite domain of dependence property of the hyperbolic operators, and to the unique continuation theorem (Tatary,1996).

In [G.Eskin, ArXiv:math.AP/050816/v.2] was considered the inverse problem for the hyperbolic equations with time-dependent coefficients of general form:

$$(16)\qquad \left(-i\frac{\partial}{\partial t}+A_0(x,t)\right)^2 u(x,t)$$
$$-\sum_{j,k=1}^{n}\frac{1}{\sqrt{g(x)}}\left(-i\frac{\partial}{\partial x_j}+A_j(x,t)\right)\sqrt{g(x)}g^{jk}(x)$$
$$\cdot\left(-i\frac{\partial}{\partial x_k}+A_k(x,t)\right)u+V(x,t)u=0,$$

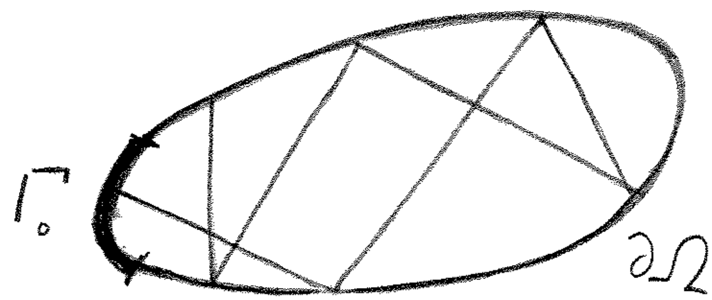
where $\|g^{jk}(x)\|^{-1}$ is the metric tensor in $\overline{\Omega}$, $g(x)=\det\|g^{jk}\|^{-1}$, $A_j(x,t),\ 0\le j\le n$, and $V(x,t)$ are smooth in $x\in\overline{\Omega}$ and real analytic in $t,\ t\in[0,T_0]$.

The result is that the D-to-N operator on $\Gamma_0\times(0,T_0)$, where Γ_0 is any open part of the boundary and T_0 is large enough, determines $\|g^{jk}(x)\|,\ A,\ V$ up to a diffeomorphism and a gauge transformation under the following assumptions:

1) A,V are analytic in t (this is required by the unique continuation theorem of Tataru and Robbiano-Zuilly.

2) The BLR (Bardos-Lebeau-Rauch) condition must be satisfied : there exists T_* such that any broken geodesics

in $\overline{\Omega}$ must intersect Γ_0 in time $t < T_*$.



Apparently, there is no relation between the inverse hyperbolic problem and the inverse problem for the Schródinger equation in the case of the time-dependent coefficients. Note that the hyperbolic equation approach works without complications in the Yang-Mills case, i.e. when $m > 1$. In particular, one can consider the Aharonov-Bohm effect for the hyperbolic equations with Yang-Mills potentials.

Hyperbolic equations of general form.

In a new work with Jim Ralston we consider the second order hyperbolic equations with time-dependent coefficients of a general form:

$$\sum_{j,k=0}^{n} \frac{1}{\sqrt{(-1)^n g(x)}} \left(-i\frac{\partial}{\partial x_j} + A_j(x)\right) \tag{17}$$

$$\sqrt{(-1)^n g(x)} g^{jk}(x) \left(-i\frac{\partial}{\partial x_k} + A_k(x)\right) u(x) - V(x)u = 0,$$

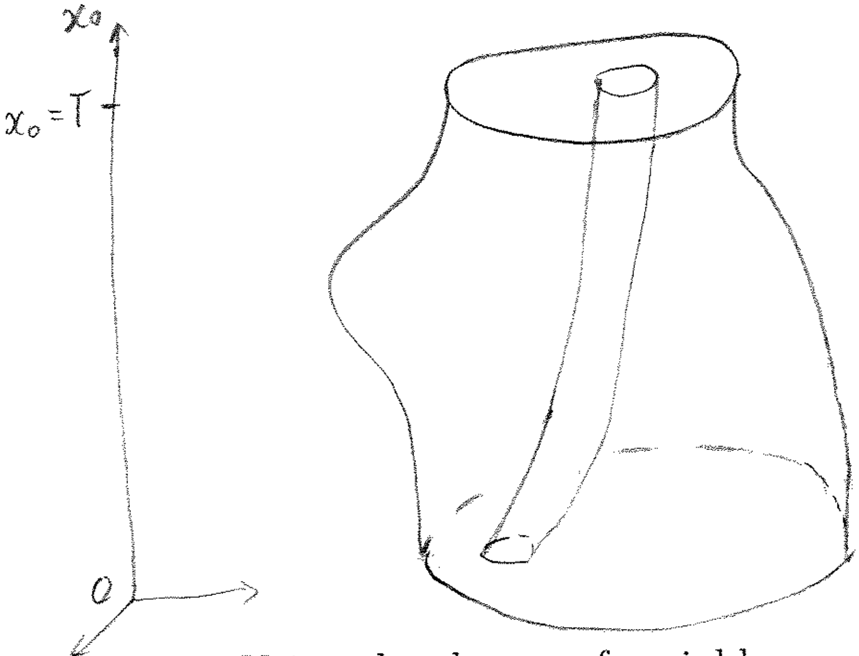
where $x = (x_0, ..., x_n) \in \mathbf{R}^{n+1}$, the quadratic form

$$\sum_{j,k=0}^{n} g^{jk}(x)\xi_j\xi_k, \qquad g^{jk}(x) = g^{kj}(x), \tag{18}$$

has the signature $(1, -1, ..., -1)$, $g(x) = \det \|g^{jk}\|^{-1}$.

We assume that $(1, 0, ..., 0)$ is a time-like direction for all $x \in \mathbf{R}^{n+1}$, i.e. $g^{00}(x) > 0$.

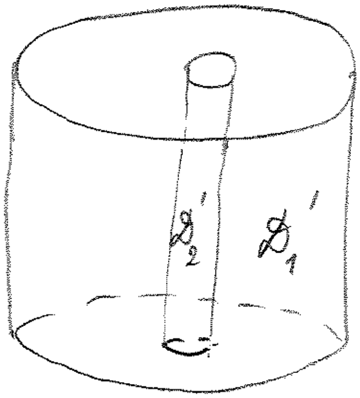
Consider a domain $D \subset \mathbf{R}^{n+1}$ of the form



Using the change of variables

$$
\begin{aligned}
y_0 &= x_0, \\
y_j &= \varphi_j(x_0, x_1, \ldots, x_n), \qquad 1 \leq j \leq n,
\end{aligned}
$$

we can transform this domain D to a simpler domain D':



$\mathcal{D}_1' = \Omega_1 \times (0,T)$

$\mathcal{D}_2' = \Omega_2 \times (0,T)$

$\mathcal{D}' = \mathcal{D}_1' \setminus \mathcal{D}_2'$

Note that equation (17) has the same form in y-coordinates. We assume as above that

1) Coefficients of the equations are real analytic in y_0.

2) The BLR-condition is satisfied.

3) We also impose the following condition : For each $y \in \overline{D}'$ the cone $\sum_{j,k=0}^{n} \eta_j \eta_k = 0$ intersects the plane $\eta_0 = 0$ only when $\eta_1 = \ldots = \eta_n = 0$.

We consider the initial - boundary value problem with zero initial conditions and zero conditions on $\partial\Omega_2 \times (0,T)$. One can show that under conditions 1), 2), 3) and T is large enough the Dirichlet-to-Neumann operator given on $\partial\Omega_1 \times (0,T)$ determines the coefficients of the hyperbolic operator up to a diffeomorphism and a gauge transformation.

Note that the condition 3) is very important. If it is not satisfied the uniqueness of the solution of the inverse problem may fail and the phenomenon of the invisibility may take place.

The nonuniqueness and the invisibility.

a) Consider the equation

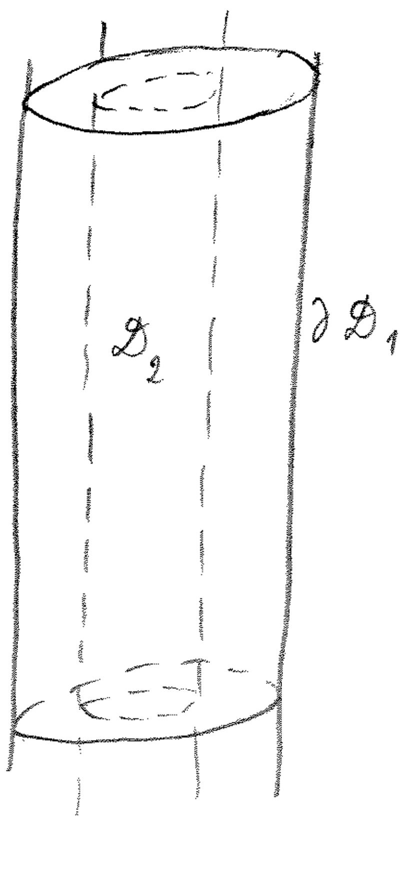
$$(1)\qquad \left(\frac{\partial}{\partial t}+a_0(x_1,x_2)\frac{\partial}{\partial r}\right)^2 u-\frac{\partial^2 u}{\partial x_1^2}-\frac{\partial^2 u}{\partial x_2^2}=0$$

in the cylinder $D_1=\{(x_1,x_2): x_1^2+x_2^2\le 1\}\times(-\infty,\infty)$. Here $a_0(x_1,x_2)\in C^\infty$, $r=\sqrt{x_1^2+x_2^2}$, $a_0(x_1,x_2)=a_0(r)$ for $r\in[\frac{1}{2}-\varepsilon,1]$, $a_0(1)=0$, $a_0'(r)<0$ on $[\frac{1}{2}-\varepsilon,1]$, $a_0(\frac{1}{2})=1$, $a_0(x_1,x_2)=0$ for $|x|<\varepsilon$.

We consider the initial-boundary value problem in D_1:

$$u(x,t)=0 \quad \text{for} \quad t<0,$$
$$u=f \quad \text{for} \quad r=1,\ t\in(-\infty,+\infty),$$

where $f=0$ for $t<0$.



Denote $D_2 = \{(x,t) : r < \frac{1}{2},\ t \in (-\infty, +\infty)\}$. For any $(x,t) \in D_2$ the domain of dependence of (x,t) is contained in D_2. Therefore $u = 0$ in D_2 for all f and all $a(x) \in C^\infty$ that coincide with $a_0(r)_0$ on $[\frac{1}{2} - \varepsilon, 1]$.

There is no uniqueness of the solution of the inverse problem in this case. Note that the domain of influence of $(x,t) \in \overline{D_1} \setminus D_2$ is contained in $\overline{D_1} \setminus D_2$.

b) Consider the equation similar to (19) with $a(x)$ changed to $-a(x)$:

$$\left(\frac{\partial}{\partial t} - a_0(x_1, x_2)\frac{\partial}{\partial r}\right)^2 u - \frac{\partial^2 u}{\partial x_1^2} - \frac{\partial^2 u}{\partial x_2^2} = 0$$

in the same domain D_1. Now for any $(x,t) \in D_2$ the domain of influence of (x,t) is contained in D_2. For any (x,t) in $\overline{D_1} \setminus D_2$ the domain of dependence of (x,t) is contained in $\overline{D_1} \setminus D_2$. Therefore the restriction of the solution of the initial-boundary value problem to $\overline{D_1} \setminus D_2$ is independent of $a(x)$ in $[\frac{1}{2} - \varepsilon, 0]$. Therefore again we have a nonuniqueness of the solution of the inverse problem.

c) We compare the cases a) and b). Note that the equations of the form (19) arise in the problems with the moving medium.

In the case a) the point in D_2 depends on D_2 only, but the point in $\overline{D_1} \setminus D_2$ depends on D_2. Therefore if we change the problem and allow a nonzero right hand sides in D_2 ("active devices") then their effect will be felt in $\overline{D_1} \setminus D_2$.

However any event in $\overline{D_1} \setminus D_2$ (for example, changing the boundary condition f) will not be felt in D_2. The domain $\{x : r < \frac{1}{2}\}$ is called sometimes a "white hole".

In the case b), even if we introduce "active devices" in D_2, the impact of any event in D_2 will be felt in D_2 only. Events in $\overline{D_1} \setminus D_2$ have impact on D_2 but not vice versa. In this case $\{x : r < \frac{1}{2}\}$ is called a "black hole".