Math 230B Homework 2

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1 Zhou

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3. Prove that if f is integrable on [0,1] and f is continuous at $\frac{1}{2}$ with $f(\frac{1}{2}) > 0$ and $f(x) \geq 0$ for $0 \leq x \leq 1$, then

$$\int_0^1 f(x)dx > 0.$$

Proof: From our first homework Spivak Question 15, if f is continuous, and $f(a) > 0$ then there exists a $\delta > 0$ such that $f(a) > 0$ for all $x \in (a-\delta, a+\delta)$. To see this let $f(a) = \epsilon > 0$. Then there exists $\delta > 0$ such that if $|x-a| < \delta$ then $|f(x) - f(a)| < \epsilon = f(a)$ since f is continuous. Then $0 < f(x) < 2f(a)$. So in this case $a = \frac{1}{2}$. Then we get that

$$\int_0^{a-\delta} f(x)dx \geq 0 \qquad \int_{a+\delta}^1 f(x)dx \geq 0$$

and we know

$$\int_{a-\delta}^{a+\delta} f(x)dx > 0$$

because $f(x) > 0$ for all $x \in (a-\delta, a+\delta)$. Putting this all together we get

$$\int_0^1 f(x)dx = \int_0^{a-\delta} f(x)dx + \int_{a-\delta}^{a+\delta} f(x)dx + \int_{a+\delta}^1 f(x)dx \geq \int_{a-\delta}^{a+\delta} f(x)dx > 0$$

So we have the desired conclusion that

$$\int_0^1 f(x)dx > 0$$

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4. Let f be continuous on $\mathbb{R}$. Let $\delta > 0$ and let

$$F(x) = \frac{1}{2\delta}\int_{-\delta}^{\delta} f(x+t)dt.$$

(a) Prove F has a continuous derivative on $\mathbb{R}$.

Proof: First we show that $F'(a)$ exists for all a. So look at the definition of $F'(a)$

$$\lim_{x\to a} \frac{F(x)-F(a)}{x-a} = \lim_{x\to a} \frac{\frac{1}{2\delta}\left(\int_{-\delta}^{\delta} f(x+t) - f(a+t)dt\right)}{x-a}$$

Now use a change of variables to make the problem a little easier to understand. Let $u = x+t$ and $z = a+t$. Then this changes to

$$\lim_{x\to a} \frac{\frac{1}{2\delta}\left(\int_{-\delta}^{\delta} f(x+t) - f(a+t)dt\right)}{x-a} = \lim_{u\to z} \frac{\frac{1}{2\delta}\left(\int_{-\delta}^{\delta} f(u) - f(z)du\right)}{(u-t)-(z-t)} = \lim_{u\to z} \frac{1}{2\delta}\int_{-\delta}^{\delta} \frac{f(u)-f(z)}{u-z}du$$

Since f is continuous the limit of the integral is integral of the limit. So we can continue to simplify

$$\lim_{u\to z} \frac{1}{2\delta}\int_{-\delta}^{\delta} \frac{f(u)-f(z)}{u-z}du = \frac{1}{2\delta}\int_{-\delta}^{\delta} \lim_{u\to z} \frac{f(u)-f(z)}{u-z}du = \frac{1}{2\delta}\int_{-\delta}^{\delta} f'(u)du$$

Now switch back to a to get

$$\frac{1}{2\delta}\int_{-\delta}^{\delta} f'(u)du = \frac{1}{2\delta}\int_{-\delta}^{\delta} f'(a+t)dt = \frac{1}{2\delta}\Bigg[f(a+\delta) - f(a-\delta)\Bigg].$$

So $F'(a)$ exists and it equals $\frac{1}{2\delta}\Bigg[f(a+\delta) - f(a-\delta)\Bigg]$. ************************I think we can say that since $\frac{1}{2\delta}\Bigg[f(a+\delta) - f(a-\delta)\Bigg]$ is a continuous function that $F'(a)$ is continuous as well.

(b) Given a closed interval $[a,b]$ and $\epsilon > 0$, show that it is possible to choose $\delta > 0$ so that $|F(x) - f(x)| < \epsilon$ for all $x \in [a,b]$.

Proof: Since $f(x) = \frac{1}{2\delta}\int_{-\delta}^{\delta} f(x)dt$ we can say that

$$|F(x) - f(x)| = \left|\frac{1}{2\delta}\int_{-\delta}^{\delta} f(x+t) - f(x)dt\right|$$

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5. Let f be Riemann integrable function on the closed interval [0,1] which satisfies $0 \le f(x) \le M$ for all $0 \le x \le 1$. Prove that

$$\lim_{x\to 0^+} x\int_x^1 \frac{f(t)}{t}dt = 0$$

Proof: Since $f(x)$ is between 0 and M we can bound the integral like so

$$\lim_{x\to 0^+} x\int_x^1 \frac{f(t)}{t}dt \le \lim_{x\to 0^+} x\int_x^1 \frac{M}{t}dt = \lim_{x\to 0^+} Mx\int_x^1 \frac{1}{t}dt = \lim_{x\to 0^+} Mx[\ln(1) - \ln(x)]$$

$$= \lim_{x\to 0^+} -Mx\ln x = M\lim_{x\to 0^+}\frac{-\ln(x)}{\frac{1}{x}} = M\lim_{x\to 0^+}\frac{-\frac{1}{x}}{-\frac{1}{x^2}} = M\lim_{x\to 0^+} x = 0$$

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6. Suppose that f is continuous and $\lim_{x\to\infty} f(x) = a$. Prove that

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x f(t)dt = a.$$

Proof: Since f is continuous it is integrable, so let $F(x) = \int_0^x f(t)dt$. Then

$$\lim_{x\to\infty}\frac{1}{x}\int_0^x f(t)dt = \lim_{x\to\infty}\frac{F(x)}{x} = \lim_{x\to\infty}\frac{F'(x)}{1} = \lim_{x\to\infty} f(x) = a$$

where we used l'Hospital's rule in the second equality. *********Maybe I have to show that $\lim_{x\to\infty} F(x) = \infty$

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7. Find a continuous function f with $f(x) \geq 0$ for all x such that $\int_0^\infty f(x)dx$ exists, but $\lim_{x\to\infty} f(x)$ does not exist.

Proof: Consider the function

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ g_n(x) & n \leq x \leq n+1 \end{cases}$$

where

$$g_n(x) = \begin{cases} 2^n(x-(n-1)) & \text{if } n-1 \leq x \leq n-1+\frac{1}{2^{n+1}} \\ -2^n(x-\frac{1}{2^n}) & n-1+\frac{1}{2^{n+1}} < x \leq n-1+\frac{1}{2^n} \\ 0 & n-1+\frac{1}{2^n} < x \leq n \end{cases}$$

for $n = 1, 2, 3, \ldots$. This function is very hard to explain in mathematical notation but easy to understand if you draw it:

So if we look at this function, it is infinitely often zero and also infinitely often 1. So $\lim_{x\to\infty} f(x)$ does not exist. But if we look at $\int_0^\infty f(x)dx$ we get that it is equal to $\sum_{i=1}^\infty \frac{1}{2^i} = 1$ so the indefinite integral actually exists and equals 1.

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8. Let f be integrable on $[a,b]$, and define

$$F(x) = \int_a^x f(t)dt, \qquad x \in [a,b],$$

then F is continuous on $[a,b]$. Further, if $c \in (a,b)$ and f is continuous at c, then F is differentiable at c and $F'(c) = f(c)$.

Proof:

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9. (a) Prove that the function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ rational} \\ 1 & \text{otherwise} \end{cases}$$

is not Riemann integrable on [0,1].

Proof: Given any partition P

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i.$$

Since $x_{i-1} < x_i$ there will always be an irrational value and a rational value between x_{i-1} and x_i which means M_i will always be 1 and m_i will always be 0. So for any partition P

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} \Delta x_i = 1$$

which means the function does not satisfy the criterion for Riemann Integrability.

(b) Prove

$$f(x) = \begin{cases} 0 & \text{if } x \text{ irrational or } 0 \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in lowest terms, } q > 0 \end{cases}$$

is Riemann integrable on [0,1].

Proof: We want to show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $||P|| < \delta$ then $U(P,f) - L(P,f) < \epsilon$. If we pick $P = \{x_0, x_1, ...x_n\}$ and let $\delta = \epsilon/n$ be a partition of [0,1] such that $||P|| < \delta = \epsilon/n$

$$U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i < \sum_{i=1}^{n} (M_i - m_i)\delta.$$

As well, we know that $M_i \leq 1$ and $m_i = 0$ for any sub-interval. So the sum reduces to

$$U(P,f) - L(P,f) < \sum_{i=1}^{n} (M_i - m_i)\delta \leq \sum_{i=1}^{n} \epsilon/n = \epsilon$$

Given $\epsilon > 0$ look at the line of height $y = \epsilon/2$. By Archimedes, $\exists N$ such that $q > N$ $\frac{1}{q} < \epsilon 2 \Rightarrow \epsilon q > 2$. At most a finite number say L points are above the line $\epsilon/2$. Choose mesh $< \frac{\epsilon}{4L}$ for partition P.

$$U(f,P) \leq 2L(1)(\frac{\epsilon}{4L}) + \epsilon/2$$

$$L(f,P) = 0$$

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10. Suppose that f is bounded on $[a,b]$ and that f is continuous at each point in $[a,b]$ with the exception of $x_0 \in (a,b)$. Prove that f is Riemann integrable on $[a,b]$.

Proof:

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11. Let $f(x)$ be a differentiable function defined on the closed interval [0,1] and such that $|f'(x)| \leq M$. Prove that

$$\left| \int_0^1 f(x)dx - \frac{1}{n}\sum_{k=1}^{n} f(\frac{k}{n}) \right| \leq \frac{M}{n}$$

for n = 1, 2 , 3, ...

Proof:

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12. Let f be monotone increasing on $[a,b]$ and suppose that f is discontinuous at $c \in (a,b)$. Show that $F(x) = \int_a^x f$ is not differentiable at $x = c$.

Proof: Suppose that F is differentiable at $x = c$. Then we know that

$$F'(c) = \lim_{x \to c} \frac{F(x) - F(c)}{x - c} = \lim_{x \to c} \frac{\int_a^x f - \int_a^c f}{x - c} = \lim_{x \to c} \frac{-\int_x^c f}{x - c} = \lim_{x \to c} \frac{\int_c^x f}{x - c}$$

. If we let x approach c from the right then by the Intermediate Value Theorem for integrals we can a sequence $\{x_{c_n}\} \to c$ and $c < x_{c_n} < x$ for all x and $f(x_{c_n}) = \frac{\int_c^x f}{x-c}$. Then if we let $x \to c$ then we will have $x_{c_n} \to c$ as well and we will have

$$\lim_{x \to c} f(x_{c_n}) = \lim_{x \to c} \frac{\int_c^x f}{x - c} = F'(c) = f(c)$$

This means f is continuous at c which is a contradiction. ****************So we have that as $\{x_{c_n}\} \to c, f(x_{c_n}) \to F'(c)$.

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13. Suppose that f and g are positive and continuous on $I = [a,b]$. Prove that there is a number $\xi \in I$ so that

$$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$$

Proof: p. 274 question 23

$$\int_a^b f(x)dx = f(\xi)(b-a)$$

As well

$$\int_a^b f(x)g(x)dx = f(z)g(z)(b-a)$$

Putting these two together we get

$$\int_a^b f(x)g(x)dx = f(z)g(z)\frac{\int_a^b f(x)dx}{f(\xi)}$$

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14. Prove that if f is integrable on $[a,b]$, then for any $\epsilon > 0$ there is a continuous function $g \leq f$ with $\int_a^b f - \int_a^b g < \epsilon$.

Proof:

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15. Suppose that f' is integrable on [0,1] and $f(0) = 0$. Prove that for all x in [0,1] we have

$$|f(x)| \leq \sqrt{\int_0^1 |f'|^2}.$$

Proof: We have that $|\int_a^b f| \leq \int_a^b |f|$. So using the Cauchy Schwarz Inequality **********************

$$|f(x)| = \left|\int_0^1 f'(x)dx\right| \leq \int_0^1 |f'(x)|dx \leq \left(\int_0^1 |f'|^2\right)^{\frac{1}{2}}\left(\int_0^1 1^2\right)^{\frac{1}{2}} = \sqrt{\int_0^1 |f'|^2}$$

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16. If f is continuous on $[a,b]$ and ϕ is bounded and increasing on $[a,b]$ there exists $\xi \in [a,b]$ such that

$$\int_a^b f d\phi = f(\xi)[\phi(b) - \phi(a)].$$

Proof: We showed in class that if f is continuous on $[a,b]$ and $\alpha \in BV[a,b]$ then (RS)$\int_a^b f d\alpha = [\sup f(x)]V(\alpha,[a,b])$. Since f is continuous on a compact set, i.e. an interval in $\mathbb{R}$ then it attains its maximum on $[a,b]$. So there exists $\xi \in [a,b]$ such that $f(\xi) \geq f(x)$ for all $x \in [a,b]$ and $V(\phi,[a,b]) = \phi(b) - \phi(a)$. So we have that (RS)$\int_a^b f d\phi = f(\xi)[\phi(b) - \phi(a)]$.

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26. A function s defined on $[a,b]$ is called a **step function** if there is a partition $P = \{t_0, ..., t_n\}$ of $[a,b]$ such that s is a constant on each (t_{i-1}, t_i) (the values of s at t_i may be arbitrary).

(a) Prove that if f is integrable on $[a,b]$, then for any $\epsilon > 0$ there is a step function $s_1 \leq f$ with $\int_a^b f - \int_a^b s_1 < \epsilon$, and also a step function $s_2 \geq f$ with $\int_a^b s_2 - \int_a^b f < \epsilon$.

Proof: Let the step function be equal to the min on each partition then s_1 will be equal to the lower sum. Since $\int_a^b f - L < U - L < \epsilon$ and $s_1 = L$ so $\int -s_1 < \epsilon$

(b) Suppose that for all $\epsilon > 0$ there are step functions $s_1 \leq f$ and $s_2 \geq f$ such that $\int_a^b s_2 - \int_a^b s_1 < \epsilon$. Prove f is integrable. ********Check question********

Proof: Since we chose $s_1 = L$ and $s_2 = U$ then we have $U - L < \epsilon$ which means f is integrable.

(c) Find a function f which is not a step function, but which satisfies $\int_a^b f = L(f,P)$ for some partition P of $[a,b]$.

Proof: Let $f(x) = 1$ except at 1 point it is 2

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27. Prove that if f is integrable on $[a,b]$, then for any $\epsilon > 0$ there are continuous functions $g \leq f \leq h$ with $\int_a^b h - \int_a^b g < \epsilon$. Hint: First get step functions with this property, and then continuous ones. A picture will help immensely.

Proof:

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30. The purpose of this problem is to show that if f is integrable on $[a,b]$, then f must be continuous at many points in $[a,b]$

(a) Let $P = \{t_0, ..., t_n\}$ be a partition of $[a,b]$ with $U(f,P) - L(f,P) < b - a$. Prove that for some i we have $M_i - m_i < 1$.

Proof: Given P above, we know that $U(f,P) - L(f,P) < b - a$. So then:

$$U(f,P) - L(f,P) = \sum_{i=1}^{n} M_i \Delta t_i - \sum_{i=1}^{n} m_i \Delta t_i = \sum_{i=1}^{n} (M_i - m_i) \Delta t_i < b - a$$

Since there are only finitely many division points we can let $\mu_k = \min\{M_i - m_i : 1 \leq i \leq n\}$. Then we get:

$$\sum_{i=1}^{n} (\mu_k) \Delta t_i = \mu_k \sum_{i=1}^{n} \Delta t_i < b - a$$

$$\Rightarrow \mu_k (b - a) < b - a$$

$$\Rightarrow \mu_k < 1$$

So there exists an $i = k$ such that $\mu_i = M_i - m_i < 1$.

(b) Prove that there are numbers a_1 and b_1 with $a < a_1 < b_1 < b$ and $\sup\{f(x) : a_1 \leq x \leq b_1\} - \inf\{f(x) : a_1 \leq x \leq b_1\} < 1$. (You can choose $[a_1, b_1] = [t_{i-1}, t_i]$ from part (a) unless $i = 1$ or n; and in these two cases a very simple device solves the problem.)

Proof: We proved above that there exists $[a_1, b_1] = [t_{i-1}, t_i]$ such that $a < a_1 < b_1 < b$ and $\sup\{f(x) : a_1 \leq x \leq b_1\} - \inf\{f(x) : a_1 \leq x \leq b_1\} < 1$. But if $i = 1$ or n we do not have that $a < a_1$ or $b_1 < b$. But if this is the case then we can let $a_1 = a + \delta$ or $b_1 = b - \delta$. Take the case when $i = 1$. Moving in a small amount δ will still ensure that $\sup\{f(x) : a_1 \leq x \leq b_1\} - \inf\{f(x) : a_1 \leq x \leq b_1\} < 1$ because $\sup\{f(x) : a_1 = a + \delta \leq x \leq b_1\} \leq \sup\{f(x) : a \leq x \leq t_1\}$. Taking away points can only make the supremum of a set smaller. And similarly for the infimum we get that $\inf\{f(x) : a_1 = a + \delta \leq x \leq b_1\} \geq \inf\{f(x) : a \leq x \leq t_1\}$ because taking away points can only make the supremum bigger. So then we have that $\sup\{f(x) : a_1 = a + \delta \leq x \leq b_1\} - \inf\{f(x) : a_1 = a + \delta \leq x \leq b_1\} \leq \sup\{f(x) : a \leq x \leq b_1\} - \inf\{f(x) : a \leq x \leq b_1\} < 1$. The same argument follows for $i = n$.

(c) Prove that there are numbers a_2 and b_2 with $\sup\{f(x) : a_1 \leq x \leq b_1\} - \inf\{f(x) : a_1 \leq x \leq b_1\} < \frac{1}{2}$

Proof:

(d) Continue in this way to find a sequence of intervals $I_n = [a_n, b_n]$ such that $\sup\{f(x) : x \in I_n\} - \inf\{f(x) : x \in I_n\} < \frac{1}{n}$. Apply the Nested Intervals Theorem to find a point x at which f is continuous.

Proof:

(e) Prove that f is continuous at infinitely many points in $[a, b]$.

Proof:

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31. Recall, from Problem 13, that $\int_a^b f \geq 0$ if $f(x) \geq 0$ for all $x \in [a, b]$

(a) Give an example where $f(x) \geq 0$ for all x, and $f(x) > 0$ for some x in $[a, b]$, and $\int_a^b f = 0$.

Proof:

(b) Suppose $f(x) \geq 0$ for all x in $[a, b]$ and f is continuous at x_0 in $[a, b]$ and $f(x_0) > 0$. Prove that $\int_a^b f > 0$. Hint: It suffices to find one lower sum $L(f, P)$ which is positive.

Proof:

(c) Suppose f is integrable on $[a, b]$ and $f(x) > 0$ for all x in $[a, b]$. Prove that $\int_a^b f > 0$. Hint: You will need Problem 30; indeed that was one reason for including Problem 30.

Proof:

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32. (a) Suppose that f is continuous on $[a, b]$ and $\int_a^b fg = 0$ for all continuous functions g on $[a, b]$. Prove that $f = 0$. (This is easy: there is an obvious g to choose.)

Proof: I think if you pick $g = |f|$ then it will work.

(b) Suppose f is continuous on $[a, b]$ and that $\int_a^b fg = 0$ for those continuous functions g on $[a, b]$ which satisfy the extra conditions $g(a) = g(b) = 0$. Prove that $f = 0$. Hint: Derive a contradiction from the assumption $f(x_0) > 0$ or $f(x_0) < 0$; the g you pick will depend on the behavior of f near x_0.

Proof:

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34. Let $f(x) = 0$ for irrational x, and $1/q$ if $x = p/q$ in lowest terms. Show that f is integrable on $[0, 1]$ and that $\int_0^1 f = 0$. (Every lower sum is clearly 0; you must figure out how to make upper sums small.)

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9. Let

$$f(x) = \begin{cases} \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & x = 0 \end{cases}$$

Is the function $F(x) = \int_0^x f$ differentiable at 0?

Proof:

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22. Suppose that f' is integrable on [0,1] and $f(0) = 0$. Prove that for all x in [0,1] we have

$$|f(x)| \leq \sqrt{\int_0^1 |f'|^2}.$$

Show why the hypothesis $f(0) = 0$ is needed. Hint: Problem 13-39.

Proof: Using the fact that $|\int_a^b f| \leq \int_a^b |f|$ and 13-39

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29. (a) If f is continuous on [0,1], compute

$$\lim_{x \to 0^+} x \int_x^1 \frac{f(t)}{t} dt.$$

Proof: Since f is continuous on $[0, 1]$ it is bounded by some number, call it M. We can use this to say that

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Same as the question from Zhou's Problems above

(b) If f is integrable on [0,1] and $\lim_{x \to 0} f(x)$ exists, compute

$$\lim_{x \to 0^+} x \int_x^1 \frac{f(t)}{t^2} dt.$$

3 Rudin

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1. Suppose α increases on $[a, b]$, $a \leq x_0 \leq b, \alpha$ is continuous at $x_0, f(x_0) = 1, f(x) = 0$ if $x \neq x_0$. Prove that $f \in \mathcal{R}(\alpha)$ and that $\int f d\alpha = 0$.

Proof: Suppose $P = \{x_0, x_1, ..., x_n\}$ is a partition of $[a, b]$ where $x_{k-1} < 1 < x_k$ for some k. Then $M_i = m_i$ for every $i \neq k$ and $M_k = 1, m_k = 0$ on the interval $[x_{k-1}, x_k]$. Then

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^{n}(M_i - m_i)\Delta\alpha_i = (M_k - m_k)(\alpha(x_k) - \alpha(x_{k-1})) = \alpha(x_k) - \alpha(x_{k-1})$$

Since α is continuous, we know that for $\epsilon > 0$ there exists a δ such that if $|x_k - x_{k-1}| < \delta$ then $|\alpha(x_k) - \alpha(x_{k-1})| < \epsilon$. So given $\epsilon > 0$ we can find a δ such that if $|x_k - x_{k-1}| < \delta$ then $U(P, f, \alpha) - L(P, f, \alpha) = \alpha(x_k) - \alpha(x_{k-1}) < \epsilon$ which means $f \in \mathcal{R}(\alpha)$. Since f is integrable we know that $L(P, f, \alpha) \leq \int f d\alpha \leq U(P, f, \alpha)$ and since $L(P, f, \alpha) \leq 0 \leq U(P, f, \alpha)$ it must be the case that $\int f d\alpha = 0$.

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2. Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$. (Compare this with Exercise 1).

Proof: Assume that instead $f(x) \neq 0$ for some x. Then there exists some x_0 such that $f(x_0) > 0$. From our first homework Spivak Question 15, if f is continuous, and $f(x_0) > 0$ then there exists a $\delta > 0$ such that $f(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. To see this let $f(x_0) = \epsilon > 0$. Then there exists $\delta > 0$ such that if $|x - x_0| < \delta$ then $|f(x) - f(x_0)| < \epsilon = f(x_0)$ since f is continuous. Then $0 < f(x) < 2f(x_0)$. Then we get that

$$\int_0^{x_0-\delta} f(x)dx \geq 0 \qquad \int_{x_0+\delta}^1 f(x)dx \geq 0$$

and we know

$$\int_{x_0-\delta}^{x_0+\delta} f(x)dx > 0$$

because $f(x) > 0$ for all $x \in (x_0 - \delta, x_0 + \delta)$. Putting this all together we get

$$\int_0^1 f(x)dx = \int_0^{x_0-\delta} f(x)dx + \int_{x_0-\delta}^{x_0+\delta} f(x)dx + \int_{x_0+\delta}^1 f(x)dx \geq \int_{x_0-\delta}^{x_0+\delta} f(x)dx > 0$$

This contradicts the fact that $\int_a^b f(x)dx = 0$ so it must be the case that $f(x) = 0$ for all $x \in [a, b]$.

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3. Define three functions $\beta_1, \beta_2, \beta_3$ as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j(x) = 1$ if $x > 0$ for $j = 1, 2, 3$; and $\beta_1(0) = 0, \beta_2(0) = 1, \beta_3(0) = \frac{1}{2}$. Let f be bounded function on [-1,1].

(a) Prove that $f \in \mathcal{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then

$$\int f d\beta_1 = f(0).$$

Proof: ($\Rightarrow$) Assume $f \in \mathcal{R}(\beta_1)$. Then for all $\epsilon > 0$ there exists a P such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. So

$$U(P, f, \beta_1) - L(P, f, \beta_1) = \sum_{i=1}^{n} (M_i - m_i)\Delta\beta_1$$

any partition that does include 0 as a division point will have no problem from the right but from the left it will always be zero. So that is why the limit from the right exists.

(b) State and prove a similar result for β_2.

Proof:

(c) Prove that $f \in \mathcal{R}(\beta_3)$ if and only if f is continuous at 0.

Proof:

(d) If f is continuous at 0 prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0).$$

Proof:

---

4. If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathcal{R}$ on $[a, b]$ for any $a < b$.

Proof: Given any partition P

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i$$

. Since $x_{i-1} < x_i$ there will always be an irrational value and a rational value between x_{i-1} and x_i which means M_i will always be 1 and m_i will always be 0. So for any partition P

$$U(P, f) - L(P, f) = \sum_{i=1}^{n} \Delta x_i = b - a$$

which means the function does not satisfy the criterion for Riemann Integrability.

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5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathcal{R}$ on $[a, b]$. Does it follow that $f \in \mathcal{R}$? does the answer change if we assume that $f^3 \in \mathcal{R}$?

Proof: Consider the function that is 1 if x is irrational and -1 if x is rational. This is obviously not integrable by the same argument as the one from Exercise 4. But $f^2 = 1$ for all x. This function is obviously integrable on any interval $[a,b]$. So it doesn't hold that any bounded function f with $f^2 \in \mathcal{R}$ is itself in $\mathcal{R}$.

Since f is bounded and $\phi(x) = \sqrt[3]{x}$ is a continuous function on any interval $[m,M]$ we can say that $h(x) = \phi(f(x)^3) = f(x)$ is in $\mathcal{R}$ on $[a,b]$ by Theorem 6.11 of Rudin. The reason we can't use the same trick for f^2 is because the square root function does not give back f but instead gives back $|f|$.

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6. Let P be the Cantor set constructed in Sec. 2.44. Let f be a bounded real function on [0,1] which is continuous at every point outside P. Prove that $f \in \mathcal{R}$ on [0,1]. Hint: P can be covered by finitely many segments whose total length can be made as small as desired. Proceed as in Theorem 6.10.

Proof:

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7. Suppose f is a real function on (0,1] and $f \in \mathcal{R}$ on $[c,1]$ for every $c > 0$. Define

$$\int_0^1 f(x)dx = \lim_{c\to 0} \int_c^1 f(x)dx$$

if this limit exists (and is finite).

(a) If $f \in \mathcal{R}$ on [0,1], show that this definition of the integral agrees with the old one.

Proof: If $f \in \mathcal{R}$ on [0,1] then

$$\int_c^1 f(x)dx = \int_0^1 f(x)dx - \int_0^c f(x)dx$$

and f is bounded. If $|f(x)| \leq M$ then

$$\left| \int_0^c f(x)dx \right| \leq Mc$$

This means $\lim_{c\to 0} |\int_0^c f(x)dx| \leq \lim_{c\to 0} Mc = 0$. So now if we look at the first equality as $c \to 0$

$$\lim_{c\to 0} \int_c^1 f(x)dx = \lim_{c\to 0} \left[\int_0^1 f(x)dx - \int_0^c f(x)dx \right] = \int_0^1 f(x)dx$$

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.

Proof:

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8. Suppose $f \in \mathcal{R}$ on $[a,b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely*.

Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$. Prove that

$$\int_1^\infty f(x)dx$$

converges if and only if

$$\sum_{n=1}^\infty f(n)$$

converges. (This is the so-called "integral test" for convergence of series.)

Proof: ($\Rightarrow$) Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1, \infty)$ and $\int_1^\infty f(x)dx$ converges. Consider the partition $P_n = \{x_1, x_2, ...x_n\}$ where $x_i = i$ (note that there is no x_0 for ease of notation). Since f is decreasing monotonically on $[1, n]$ for all $n \in \mathbb{N}$ we can say the maximum value M_i on the interval occurs at the left endpoint and the minimum m_i occurs at the right endpoint. So we know then that

$$L(P_n, f) \leq R(P_n, f) \leq U(P_n, f)$$

$$\sum_{i=1}^n m_i \Delta x_{i+1} \leq \sum_{i=1}^n f(\xi_i)\Delta x_{i+1} \leq \sum_{i=1}^n M_i \Delta x_{i+1}$$

Since m_i occurs at the right endpoint as stated above, the area under $f(x)$ from $[x_i, x_{i+1}]$ is less than or equal to $m_i(x_{i+1} - x_i) = m_i$. As well the area under $f(x)$ from $[x_i, x_{i+1}]$ is greater than or equal to M_i. So this gives us

$$\sum_{i=1}^n m_i \leq \sum_{i=1}^n f(\xi_i) \leq \sum_{i=1}^n M_i$$

---

10. Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

Proof: If we divide through by v^q then

$$\frac{u}{v^{q-1}} \leq \frac{u^p}{v^q p} + \frac{1}{q}$$

Now let $x = u^p/v^q$ and the fact that

$$\frac{u}{v^{1-q}} = \frac{(u^p)^{\frac{1}{p}}}{v^{-\frac{p}{q}}}$$

and $q - 1 = \frac{q}{p}$ to get

$$\frac{(u^p)^{\frac{1}{p}}}{v^{-\frac{p}{q}}} = \frac{(u^p)^{\frac{1}{p}}}{(v^p)^{\frac{1}{p}}} = x^{\frac{1}{p}}$$

Now use this to transform the original equation to an equation of x:

$$\Rightarrow x^{\frac{1}{p}} \leq \frac{1}{p}x + \frac{1}{q}$$

Now consider the function

$$f(x) = \frac{1}{p}x + \frac{1}{q} - x^{\frac{1}{p}}$$

So we have reduced the problem to showing that $f(x) \geq 0$ for all x. It is easy to see that $f(0) = \frac{1}{p} > 0$ and $\lim_{x\to\infty} f(x) = \infty$. So the function starts out positive and is eventually always greater than 0. So we must examine what the function values are at any local minima and if they are positive we are done. Set the derivative

$$f'(x) = \frac{1}{p} - \frac{1}{p}x^{\frac{1}{p}-1}$$

equal to zero to get that $x^{\frac{1}{p}-1} = 1$ which is only true when $x = 1$. So $f(1) = \frac{1}{p} + \frac{1}{q} - 1 = 0$. So f is always non-negative which gives us what we wanted. As well, we can see that equality only occurs when $a = 1 = u^p/v^q \Rightarrow u^p = v^q$.

(b) If $f \in \mathcal{R}(\alpha), g \in \mathcal{R}(\alpha), f \geq 0, g \geq 0$, and

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

Proof: This follows from part (a). Since f and g fulfill the requirements of u and v above for all x then we know

$$fg \leq \frac{f^p}{p} + \frac{g^q}{q}$$

$$\Rightarrow \int_a^b fg d\alpha \leq \int_a^b \frac{f^p}{p} d\alpha + \int_a^b \frac{g^q}{q} d\alpha$$

$$\Rightarrow \int_a^b fg d\alpha \leq \frac{1}{p} + \frac{1}{q} = 1$$

This all follows from what is given and what we established in part (a).

(c) If f and g are complex functions in $\mathcal{R}(\alpha)$, then

$$\left| \int_a^b fg d\alpha \right| \leq \left\{ \int_a^b |f|^p d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q d\alpha \right\}^{1/q}$$

This is $Holder's\ inequality$. When $p = q = 2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is a very special case of this.)

Proof: If either f or g is identically zero then we have equality. So assume both are not zero. That means $\left\{ \int_a^b |f|^p d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q d\alpha \right\}^{1/q} \neq 0$. Then we can rewrite as

$$\frac{|\int_a^b fg d\alpha|}{\left\{ \int_a^b |f|^p d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q d\alpha \right\}^{1/q}} \leq \int_a^b \frac{|f||g|}{\left\{ \int_a^b |f|^p d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q d\alpha \right\}^{1/q}} d\alpha \leq 1$$

As well we have

$$\int_a^b \frac{|f|}{\left\{ \int_a^b |f|^p d\alpha \right\}^{1/p}} d\alpha = 1 = \int_a^b \frac{|g|}{\left\{ \int_a^b |g|^q d\alpha \right\}^{1/q}} d\alpha$$

*************************should probably show this, don't know how*****************************

Using part (c) it follows that

$$\int_a^b \frac{|f||g|}{\left\{ \int_a^b |f|^p d\alpha \right\}^{1/p} \left\{ \int_a^b |g|^q d\alpha \right\}^{1/q}} d\alpha \leq 1$$

which is what we wanted to show.

(d) Show that Holder's inequality is also true for the "improper" integrals described in Exercies 7 and 8.

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11. Let α be a fixed increasing function on $[a, b]$. For $u \in \mathcal{R}(\alpha)$, define

$$||u||_2 = \left\{ \int_a^b |u|^2 d\alpha \right\}^{1/2}.$$

Suppose $f, g, h \in \mathcal{R}(\alpha)$, and prove the triangle inequality

$$||f - h||_2 \leq ||f - g||_2 + ||g - h||_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Proof: Begin with $||u + v||_2^2$ because it is easier to manipulate than $||u + v||_2$. We get

$$\begin{aligned} ||u+v||_2^2 &= \int_a^b |u+v|^2 d\alpha \\ &= \int_a^b |u|^2 d\alpha + 2\int_a^b |u||v| d\alpha + \int_a^b |v|^2 d\alpha \\ &\leq \int_a^b |u|^2 d\alpha + 2\Big(\int_a^b |u|^2 d\alpha\Big)^{\frac{1}{2}} \Big(\int_a^b |v|^2 d\alpha\Big)^{\frac{1}{2}} d\alpha + \int_a^b |v|^2 d\alpha \\ &= \bigg(\Big(\int_a^b |u|^2 d\alpha\Big)^{\frac{1}{2}} + \Big(\int_a^b |v|^2 d\alpha\Big)^{\frac{1}{2}}\bigg)^2 \end{aligned} \tag{1}$$

The inequality coming from the Exercise 10 (c) above where $p = q = 2$. Square rooting both sides of the above inequality yields

$$||u + v||_2 \leq ||u||_2 + ||v||_2$$

which is what we wanted to prove.