Math 230B Homework 2

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1. Let $f : [a, b] \to R$ be a function of bounded variation on $[a, b]$ and define a function V on $[a, b]$ by $V(a) = 0$ and $V(x) = V(f, [a, x])$ for all $x \in (a, b]$.

(a) Let $a \leq x < y \leq b$. Prove that $V(y) - V(x) = V(f, [x, y])$.

Proof: From class we have that $V(f, [a, y]) = V(f, [a, x]) + V(f, [x, y]) \Rightarrow V(f, [a, y]) - V(f, [a, x]) = V(f, [x, y]) \Rightarrow V(y) - V(x) = V(f, [x, y])$.

(b) Prove that V is increasing on $[a, b]$.

Proof: From part (a) we have that for all $x < y, V(y) - V(x) = V(f, [x, y])$ because$V(f, [x, y])$ is the supremum over all possible partitions P and $P = \{x, y\}$ is a partition. So this gives us $V(f, [x, y]) \geq |f(y) - f(x)| \geq 0 \Rightarrow V(y) - V(x) \geq 0 \Rightarrow V(y) \geq V(x)$ for all $y > x$. So we have that V is increasing on $[a, b]$.

(c) Suppose that V is continuous at $c \in [a, b]$. Prove that f is continuous at c.

Proof: We know that $|f(c+) - f(c-)| \leq |f(c+) - f(c)| + |f(c) - f(c-)| \leq |V(c+) - V(c-)|$. Since V is continuous we know that $V(c+) - V(c-) = 0 \Rightarrow |f(c+) - f(c-)| \leq 0 \Rightarrow f(c+) = f(c-)$. So f is continuous at c as well.

(d) Suppose that f is continuous at $c \in [a, b]$. Prove that V is continuous at c.

Proof: Given $\epsilon > 0$, by the approximation property of the supremum we can find a $P = \{x_0, x_1, ...x_n\}$ a partition of $[a, b]$, such that $V(f, [a, b]) - \epsilon/2 < \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$. Insert x_{new} into the partition such that $0 < x_{new} - x_0 < \delta \Rightarrow |f(x_{new}) - f(x_0)| < \epsilon$. We can do this because f is continuous. Then

$$V(f, [c, b]) - \epsilon/2 < |f(x_{new}) - f(x_0)| + |f(x_1) - f(x_{new})| + \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})|$$

$$V(f, [c, b]) - \epsilon/2 < \epsilon/2 + |f(x_1) - f(x_{new})| + \sum_{i=2}^{n} |f(x_i) - f(x_{i-1})|$$

$$\Rightarrow V(f,[c.b]) - V(f,[x_{new},b]) < \epsilon$$
$$\Rightarrow V(f,[c,x_{new}]) < \epsilon$$

This means for all $\epsilon > 0$ we can find a δ such that if $x_{new} - x_c < \delta$ then $V(x_{new}) - V(c) < \epsilon$. So V is continuous.

(e) Suppose that f is continuous on $[a,b]$. Prove that f is the difference of two increasing continuous functions.

Proof: Given f is continuous on $[a,b]$, we know from part (d) that V is continuous on $[a,b]$. And, if V is continuous on $[a,b]$ we showed in class that f is the difference of two increasing functions. To see this define $f_1(x) = V(x)$. Then in part (b) we showed this function is monotone increasing. Now let $f_2(x) = V(x) - f(x)$. Then $f_2(x)$ is also montone increasing because $x_2 \geq x_1$,

$$V(x_2) - V(x_1) = V(f,[x_1,x_2]) \geq |f(x_2) - f(x_1)| \geq f(x_2) - f(x_1)$$

$$\Rightarrow V(x_2) - f(x_2) \geq V(x_1) - f(x_1)$$
$$\Rightarrow f_2(x_2) \geq f_2(x_1)$$

So then $f_1(x) - f_2(x) = V(x) - (V(x) - f(x)) = f(x)$ and we are done.

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2. Let f be a function of bounded variation on $[a,b]$. Prove that f is the difference of two strictly increasing functions.

Proof: We know from class that we can write any function of bounded variation as the difference of two monotone increasing functions. So say that f_1, f_2 are such that $f(x) = f_1(x) - f_2(x)$. Then take any function g that is strictly increasing on $[a,b]$. Then we can say that $f(x) = f_1(x) + g(x) - g(x) - f_2(x) \Rightarrow f(x) = (f_1(x) + g(x)) - (f_2(x) + g(x))$. Then for $x_1 < x_2, g(x_1) < g(x_2)$ and $f_1(x_1) \leq f_1(x_2)$ which means $f_1(x_1) + g(x) < f_1(x_2) + g(x)$. So we have that $f_1(x) + g(x)$ is strictly increasing and it follows similarly for $f_2(x) + g(x)$ so we have that f is the difference of two strictly increasing functions.

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3. Let f be a function of bounded variation on $[a,b]$.

(a) Prove that f has one-sided limits at every point of $[a,b]$

Proof: Since f is of bounded variation, we know that $f(x) = f_1(x) - f_2(x)$ where f_1, f_2 are monotone increasing functions. We showed in class that if g is monotone increasing on (a,b) then $f(c+)$ and $f(c-)$ at every $c \in (a,b)$. So we know that for any $c \in (a,b)$, $f_1(c+)$ and $f_2(c+)$ exists so $f(c+) = f_1(c+) - f_2(c+)$ exists. The same goes for $f(c-)$ for all $c \in (a,b)$.

(b) Prove that the set of discontinuities of f is countable.

Proof: Since $f \in BV[a,b]$ we know that f can be written as the difference of two monotone increasing functions f_1, f_2. Let $\mathcal{D}(f_1)$ be equal to the set of all discontinuity points x of f_1, and similarly for f_2. Then from class we know that the set of discontinuity points of a monotone function is at most countable which means $\mathcal{D}(f_1)$ and $\mathcal{D}(f_2)$ are each at most countable. So $\mathcal{D}(f) \subseteq \mathcal{D}(f_1) \cup \mathcal{D}(f_2)$ is at most countable because it is a countable union of countable sets.

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4. Prove that $f(x) = 2|x| - |x-2|$ is of bounded variation on [-1, 4]. Compute $V(f, [-1, 4])$.

Proof: On the interval $[-1, 0]$ both functions are linear decreasing functions so they are actually both strictly decreasing. As well, $2|x| < |x-2|$ on this same interval, so it follows that $2|x|-|x-2| < 0$ and because it is the difference of two lines it is monotone decreasing. So we know that $V(f, [-1, 0]) = |f(0) - f(-1)| = 1$. As well, we can do the same on the intervals from $[0, 2]$ and $[2, 4]$. On these two intervals it is again monotone increasing. So $V(f, [0, 4]) = |f(4) - f(0)| = 8$. Since this function is a piecewise monotone decreasing/increasing function we know that f is of bounded variation on $[-1, 4]$ and $V(f, [-1, 4]) = 9$.

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5. Prove that the function f defined by $f(x) = 0$ if x is irrational and $f(x) = 1$ if x is rational is not of bounded variation on any interval $[a, b]$.

Proof: Given $M > 0$, construct a partition $P = \{a = x_0, x_1, x_2, ..., x_M, x_{M+1}, x_{M+2} = b\}$ of $[a, b]$, such that x_1 is any rational strictly between a and b, x_2 is any irrational strictly between x_1 and b, x_3 is any rational strictly between x_2 and b, and so forth alternating between rational and irrational numbers until we get to x_{M+1}. We are always able to pick an x_i regardless of whether it is rational or irrational because $\mathbb{Q}$ and $\mathbb{Q}^c$ are both dense in $\mathbb{R}$. So then if we look at the variation of f on this partition P we get $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$. Disregarding the first and last term in this sum, each other term in this sum will be 1 because one of the two x_i, x_{i-1} will be rational and the other will be irrational by the construction of P. So this sum will be at least as big as M which means that for any M we can construct a partition P of $[a, b]$ with $M + 2$ elements such that $\sum_{i=1}^{n} |f(x_i) - f(x_{i-1})| \geq M$.

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6. Let f be the function defined by $f(x) = \sqrt{x}\cos(\pi/x)$ for $x \neq 0$ and $f(0) = 0$. Prove that f is continuous on [0,1], but it is not of bounded variation on [0,1].

Proof: Since $\sqrt{x}$ and $\cos(\pi/x)$ are continuous on $(0, 1]$ so the product of these two functions will be continuous on that interval as well. To show that the function is continuous at $x = 0$, use the Squeeze Theorem:

$$-1 \leq \cos(\pi/x) \leq 1$$

$$\Rightarrow -\sqrt{x} \leq \sqrt{x}\cos(\pi/x) \leq \sqrt{x}$$

$$\Rightarrow \lim_{x\to 0} -\sqrt{x} \leq \lim_{x\to 0} \sqrt{x}\cos(\pi/x) \leq \lim_{x\to 0} \sqrt{x}$$

$$\Rightarrow 0 \leq \lim_{x\to 0} \sqrt{x}\cos(\pi/x) \leq 0$$

$$\Rightarrow \lim_{x\to 0} \sqrt{x}\cos(\pi/x) = 0$$

To show that this $f \notin BV[0, 1]$ consider the partition $\{1, \frac{2}{3}, \frac{1}{2}, \frac{2}{5}, \frac{1}{3}, \frac{2}{7}, \frac{1}{4}, ...\}$. We can not have a partition with an infinite number of points, but for any finite subset of this partition where we take

$2n-1$ of these points in succession, we will get a partial sum that looks like this:

$$\sum_{x=1}^{n} \frac{1}{\sqrt{x}} = \sum_{x=1}^{n} \frac{1}{x^{\frac{1}{2}}}$$

Since this series diverges as $n \to \infty$, we can make the sequence of partial sums as large as we like.

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15. (a) Let f be a function such that $|f(x)| \leq x^2$ for all x. Prove that f is differentiable at 0.

Proof: If we look at the definition of the derivative we get

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h) - 0}{h} = \lim_{h\to 0} \frac{f(h)}{h}.$$

We also have that $-x^2 \leq f(x) \leq x^2 \Rightarrow -h^2 \leq f(h) \leq h^2$. If we manipulate this to look like the derivative then we have that:

$$\lim_{h\to 0} \frac{-h^2}{h} \leq \lim_{h\to 0} \frac{f(h)}{h} \leq \lim_{h\to 0} \frac{h^2}{h}$$

$$\Rightarrow \lim_{h\to 0} -h \leq \lim_{h\to 0} \frac{f(h)}{h} \leq \lim_{h\to 0} h$$

$$\Rightarrow 0 \leq \lim_{h\to 0} \frac{f(h)}{h} \leq 0$$

$$\Rightarrow f'(0) = \lim_{h\to 0} \frac{f(h)}{h} = 0$$

by the Squeeze Theorem. So the derivative exists at $x = 0$, and $f'(0) = 0$.

(b) This result can be generalized if x^2 is replaced by $|g(x)|$, where g has what property?

Proof: We can replace x^2 by any function g such that $g(x) \geq 0$ for all x and $\lim_{x\to 0} |g(x)| = 0$ because the proof in part (a) will follow exactly the same.

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16. Let $\alpha > 1$. If f satisfies $|f(x) \leq |x|^\alpha$, prove that f is differentiable at 0.

Proof: We can see that $|x|^\alpha$ satisfies the requirements in 15(b). So it follows that f is differentiable at 0. To see this we can again do the same as we did in 15 to get that

$$\lim_{h\to 0} \frac{-h^\alpha}{h} \leq \lim_{h\to 0} \frac{f(h)}{h} \leq \lim_{h\to 0} \frac{h^\alpha}{h}$$

$$\Rightarrow \lim_{h\to 0} -h^{\alpha-1} \le \lim_{h\to 0} \frac{f(h)}{h} \le \lim_{h\to 0} h^{\alpha-1}$$

$$\Rightarrow 0 \le \lim_{h\to 0} \frac{f(h)}{h} \le 0$$

$$\Rightarrow f'(0) = \lim_{h\to 0} \frac{f(h)}{h} = 0$$

by the Squeeze Theorem.

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17. Let $0 < \beta < 1$. Prove that if f satisfies $|f(x)| \ge |x|^\beta$ and $f(0) = 0$, then f is not differentiable at 0.

Proof: Observe that $\lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$ and that $-f(x) \le |x|^\beta \le f(x)$. If we look at just the right part of the inequality and consider only values of $x = h > 0$ we can see that:

$$|x|^\beta \le f(x)$$

$$\Rightarrow h^\beta \le f(h)$$

$$\Rightarrow \frac{h^\beta}{h} \le \frac{f(h)}{h}$$

$$\Rightarrow \frac{1}{h^{1-\beta}} \le \frac{f(h)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{1}{h^{1-\beta}} \le \lim_{h\to 0} \frac{f(h)}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{1}{h^{1-\beta}} \le f'(0)$$

The limit on the left goes to ∞ as $h \to 0$, so the derivative is going to ∞ if we get closer to 0. If we do the same thing when $h < 0$ we would get that the derivative is going to $-\infty$ as we get closer to 0. So the f is obviously not differentiable at 0.

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18. Let $f(x) = 0$ for irrational x, and $1/q$ for $x = p/q$ in lowest terms. Prove that f is not differentiable at a for any a.

Proof: We have already shown that this function is not continuous when $x \in \mathbb{Q}$, which means f is not differentiable on $\mathbb{Q}$ either. So we are only left to show that f is not differentiable for any $x \in \mathcal{Q}^c$. So if $a \notin \mathbb{Q}$ and $h \to 0$ passing through rational numbers then $a + h$ is irrational and $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = 0$. Now look at $a = a_0.a_1a_2.....$, let $h_n = -0.0...0a_na_{n+1}...$ and we will have that $h_n \to 0$. So

$$\lim_{h_n\to 0} \frac{|f(a+h_n) - f(a)|}{|h_n|} = \lim_{h_n\to 0} \frac{|f(a+h_n)|}{|h_n|} \ge \frac{1}{10^n} \cdot 10^{n-1} = \frac{1}{10}$$

. This inequality holds because the denominator of $a + h_n$ in lowest term is at most 10^n and $|h_n| < 1/10^{n-1}$. Therefore $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \ne 0$ and so the limit does not exist.

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19. (a) Suppose that $f(a) = g(a) = h(a)$, that $f(x) \leq g(x) \leq h(x)$, for all x, and that $f'(a) = h'(a)$. Prove that g is differentiable at a, and that $f'(a) = g'(a) = h'(a)$.

Proof: This follows similar to other Squeeze Theorem problems. We start with the fact that

$$f(x) \leq g(x) \leq h(x)$$

$$\Rightarrow f(x) - f(a) \leq g(x) - g(a) \leq h(x) - h(a)$$

$$\Rightarrow \frac{f(x) - f(a)}{x - a} \leq \frac{g(x) - g(a)}{x - a} \leq \frac{h(x) - h(a)}{x - a}$$

$$\Rightarrow \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \leq \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \leq \lim_{x \to a} \frac{h(x) - h(a)}{x - a}$$

$$\Rightarrow f'(a) \leq \lim_{x \to a} \frac{g(x) - g(a)}{x - a} \leq h'(a)$$

Since $f'(a), h'(a)$ exist and are equal, we know that $g'(a)$ exists and $g'(a) = f'(a) = h'(a)$.

(b) Show that the conclusion does not follow if we omit the hypothesis $f(a) = g(a) = h(a)$.

Proof: The proof above obviously falls apart without this and to see why, consider the example $f(x) = x^2 + 1, g(x) = x, h(x) = -x^2 - 1$ at $a = 0$. Then we have that $f'(a) = h'(a) = 0$, but $g'(a) = 1$.

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22. (a) Suppose that f is differentiable at x. Prove that

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h}$$

Proof: First subtract and add $f(x)$ to the numerator

$$\lim_{h \to 0} \frac{f(x + h) - f(x) + f(x) - f(x - h)}{h}$$

$$= \frac{1}{2} \lim_{h \to 0} \left[\frac{f(x + h) - f(x)}{h} + \frac{f(x) - f(x - h)}{h} \right]$$

$$= \frac{1}{2} \left[\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} + \lim_{h \to 0} \frac{f(x) - f(x - h)}{h} \right]$$

$$= \frac{1}{2} [f'(x) + f'(x)]$$

$$= f'(x)$$

(b) Prove, more generally, that

$$f'(x) = \lim_{h, k \to 0^+} \frac{f(x + h) - f(x - k)}{h + k}$$

Proof: If let h be fixed and let $k \to 0^+$ and then after that let $h \to 0^+$ we get:

$$\lim_{h\to 0^+}\left(\lim_{k\to 0^+}\frac{f(x+h)-f(x-k)}{h+k}\right) = \lim_{h\to 0^+}\left(\lim_{k\to 0^+}\frac{f(x+h)-f(x)}{h}\right) = f'(x)$$

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11. Find $f'(0)$ if

$$f(x) = \begin{cases} g(x)\sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and $g(0) = g'(0) = 0$.

Proof: To find $f'(0)$, unravel the definition of the derivative at 0:

$$f'(0) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{f(0+h)-0}{h} = \lim_{h\to 0}\frac{f(h)}{h} = \lim_{h\to 0}\frac{g(h)\sin(1/h)}{h}$$

The last equality is only true because $h \neq 0$. So lets try to figure out what the last limit is. Start with:

$$\begin{gathered}
-1 \le \sin(1/h) \le 1 \\
\Rightarrow -g(h) \le g(h)\sin(1/h) \le g(h) \\
\Rightarrow -\frac{g(h)}{h} \le \frac{g(h)\sin(1/h)}{h} \le \frac{g(h)}{h} \\
\Rightarrow \lim_{h\to 0} -\frac{g(h)}{h} \le \lim_{h\to 0}\frac{g(h)\sin(1/h)}{h} \le \lim_{h\to 0}\frac{g(h)}{h} \\
\Rightarrow \lim_{h\to 0} -\frac{g(h)}{h} \le \lim_{h\to 0}\frac{g(h)\sin(1/h)}{h} \le \lim_{h\to 0}\frac{g(h)}{h} \\
\Rightarrow \left|\lim_{h\to 0}\frac{g(h)\sin(1/h)}{h}\right| \le \lim_{h\to 0}\frac{g(h)}{h} = \lim_{h\to 0}\frac{g(0+h)-g(0)}{h} = g'(0) = 0 \\
\Rightarrow \left|\lim_{h\to 0}\frac{g(h)\sin(1/h)}{h}\right| \le 0 \\
\Rightarrow \lim_{h\to 0}\frac{g(h)\sin(1/h)}{h} = 0
\end{gathered}$$

Putting this together from what we know about $f'(0)$, we get $f'(0) = 0$

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16. (a) Prove that if f is differentiable at a, then $|f|$ is also differentiable at a, provided that $f(a) \neq 0$.

Proof: Since f is differentiable at a it is continuous there as well. Since $f(a) \neq 0$, either $f(a) > 0$ or $f(a) < 0$. Assume first $f(a) > 0$. There exists some small neighborhood such that $f(a+h)$ is also greater than 0 for all h in that neighborhood. So then we have that

$$|f|'(a) = \lim_{h \to 0} \frac{|f(a+h)| - |f(a)|}{h} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = f'(a)$$

for h small enough. Now if we consider $f(a) < 0$, then there exists some small neighborhood such that $f(a+h)$ is also less than 0 for all h in that neighborhood. Then:

$$|f|'(a) = \lim_{h \to 0} \frac{|f(a+h)| - |f(a)|}{h} = \lim_{h \to 0} \frac{-f(a+h) - (-f(a))}{h} = -f'(a)$$

So we have show that as long as $f(a) \neq 0$ and if f is differentiable at a, then $|f|$ is also differentiable at a.

(b) Give a counterexample if $f(a) = 0$.

Proof: Let $f(x) = x$. Then $|f(x)| = |x|$. So then at $a = 0, f(a) = 0$. Look at the derivative of $|f|$ evaluated at $a = 0$:

$$|f|'(0) = \lim_{h \to 0} \frac{|f(0+h)| - |f(0)|}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

This limit does not exist because $\lim_{h \to 0^+} \frac{|h|}{h} = 1$ but $\lim_{h \to 0^-} \frac{|h|}{h} = -1$. Since these two are not equal, $|f|$ is not differentiable at $a = 0$.

(c) Prove that if f and g are differentiable at a, then the function $\max(f, g)$ and $\min(f, g)$ are differentiable at a, provided that $f(a) \neq g(a)$.

Proof: We can express the maximum/minimum of f and g like this:

$$\max(f, g) = \frac{f(x) + g(x) + |f(x) - g(x)|}{2} \qquad \min(f, g) = \frac{f(x) + g(x) - |f(x) - g(x)|}{2}.$$

From part (a) we know that since f and g are differentiable at a, then $|f|$ and $|g|$ are differentiable at a as well. As well, the sum and difference of differentiable functions are as well differentiable and multiplication of a constant does not change differentiability. So both the $\max(f, g)$ and $\min(f, g)$ are differentiable at a.

(d) give a counterexample if $f(a) = g(a)$.

Proof: Let $f(x) = x$ and $g(x) = -x$. These functions are both differentiable for all $x \in \mathcal{R}$. But $\max(f, g) = |x|$ is not differentiable at $x = 0$ because the derivative from the right is 1 but the derivative from the left is 1. The same is true of the $\min(f, g) = -|x|$.

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18. Suppose that $f^{(n)}(a)$ and $g^{(n)}(a)$ exist. Prove *Leibniz's formula*:

$$(f \cdot g)^{(n)}(a) = \sum_{k=0}^{n} \binom{n}{k} f^{(n)}(a) \cdot g^{(n-k)}(a)$$

Proof: Use induction. For $n = 1$ we just get the product rule:

$$(f \cdot g)'(a) = \sum_{k=0}^{1} \binom{1}{k} f^{(1)}(a) \cdot g^{(1-k)}(a) = \binom{1}{0} f'(a)g(a) + \binom{1}{1} f(a)g'(a) = f'(a)g(a) + f(a)g'(a)$$

Now, assume this is true for n and show it is true for $n + 1$:

$$(f \cdot g)^{(n+1)}(a) = \sum_{k=0}^{n+1} \binom{n+1}{k} f^{(n+1)}(a) \cdot g^{(n-k+1)}(a)$$

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24. This problem is a companion to Problem 3-6. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be given numbers.

(a) If $x_1, \ldots, x_n$ are distinct numbers, prove that there is a polynomial function f of degree $2n-1$, such that $f(x_j) = f'(x_j) = 0$ for $j \neq i$, and $f(x_i) = a_i$ and $f'(x_i) = b_i$.

Proof:

(b) Prove that there is a polynomial function f of degree $2n - 1$ with $f(x_i) = a_i$ and $f'(x_i) = b_i$ for all i.

Proof:

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24. (a) Suppose that the polynomial function $f(x) = x^n + a_{n-1}x^{n-1} + \ldots + a_0$ has exactly k critical points and $f''(x) \neq 0$ for all critical points x. Show that $n - k$ is odd.

Proof:

(b) For each n, show that there is a polynomial function f of degree n with k critical points if $n - k$ is odd.

Proof:

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25. (a) Prove that if $f'(x) \geq M$ for all x in $[a, b]$, then $f(b) \geq f(a) + M(b - a)$.

Proof: Since the derivative is defined at all $x \in [a, b]$ we know f is continuous on this interval. So we can use the Mean Value Theorem to say that there is a $c \in [a, b]$ such that

$$f(b) - f(a) = f'(c)(b - a)$$

Using that fact that $f'(c) \geq M$ we can say that

$$f(b) - f(a) \geq M(b - a)$$

$$\Rightarrow f(b) \geq f(a) + M(b - a)$$

(b) Prove that $f'(x) \leq M$ for all x in $[a,b]$, then $f(b) \leq f(a) + M(b-a)$.

Proof: Since the derivative is again defined at all $x \in [a,b]$ we know f is continuous on this interval. So we can use the Mean Value Theorem to say that there is a $c \in [a,b]$ such that

$$f(b) - f(a) = f'(c)(b-a)$$

Using that fact that $f'(c) \leq M$ we can say that

$$f(b) - f(a) \leq M(b-a)$$

$$\Rightarrow f(b) \leq f(a) + M(b-a)$$

(c) Formulate a similar theorem when $|f'(x)| \leq M$ for all x in $[a,b]$.

Proof: We can say that

$$f(b) \leq f(a) + M(b-a) \qquad f(b) \geq f(a) - M(b-a)$$

$$\Rightarrow f(a) - M(b-a) \leq f(b) \leq f(a) + M(b-a)$$
$$\Rightarrow |f(b) - f(a)| \leq M(b-a)$$

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26. Suppose that $f'(x) \geq M > 0$ for all x in $[0,1]$. Show that there is an interval of length $\frac{1}{4}$ on which $|f| \geq M/4$.

Proof: Since $f'(x)$ is always strictly positive and it exists at every $x \in [0,1]$, we can say the function is continuous and monotone increasing. So we can also use the Mean Value Theorem to say that there exists a $c_a \in (0,1)$ such that:

$$f(1) - f(0) = f'(c_a)(1-0)$$
$$\Rightarrow f(1) - f(0) \geq M$$

Break $[0,1]$ up into four pieces: [0, 1/4], [1/4, 1/2], [1/2, 3/4] and [3/4, 1]. Assume to the contrary that for each of these intervals $|f| < M/4$. Then that would mean that $[f(1/4) - f(0)] + [f(1/2) - f(1/4)] + [f(3/4) - f(1/2)] + [f(1) - f(3/4)] = f(1) - f(0) < M$ which is a contradiction. So at least one of these intervals of length $\frac{1}{4}$ must have $|f| \geq M/4$.

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27. (a) Suppose that $f'(x) > g'(x)$ for all x, and that $f(a) = g(a)$. Show that $f(x) > g(x)$ for $x > a$ and $f(x) < g(x)$ for $x < a$.

Proof: Let $h(x) = f(x) - g(x)$, then $h(a) = 0$, $h'(a) = f'(a) - g'(a) > 0$. So $h(x)$ is a monotone strictly increasing function for all x. So this means that $h(x) > 0$ for $x > a$ and $h(x) < 0$ for $x < a$. This means $f(x) - g(x) > 0$ for $x > a$ and $f(x) - g(x) < 0$ for $x < a$. And the desired conclusion follows directly from this.

(b) Show by an example that these conclusions do not follow without the hypothesis $f(a) = g(a)$.

Proof: If we let $f(x) = 2x$ and $g(x) = x$, then $f'(x) > g'(x)$ for all x but if we let $a = 2$ then $f(1) > g(1)$ even thought $1 < a$.

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31. (a) Give an example of a function f for which $\lim_{x\to\infty} f(x)$ exists, but $\lim_{x\to\infty} f'(x)$ does not exist.

Proof: Let $f(x) = 1/x \sin x$. The $\lim_{x\to\infty} f(x) = 0$, but the derivative is always 0 and also non-zero no matter how far out x goes. So the $\lim_{x\to\infty} f'(x)$ does not exist.

(b) Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ both exist, then $\lim_{x\to\infty} f'(x) = 0$.

Proof: Assume $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ both exist. Then we know that:

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \left(\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \right) = L$$

(c) Prove that if $\lim_{x\to\infty} f(x)$ exists and $\lim_{x\to\infty} f''(x)$ exists, then $\lim_{x\to\infty} f''(x) = 0$.

Proof:

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34. A function f is *Lipschitz of order* α at x if there is a constant C such that

$$(*)|f(x) - f(y)| \leq C|x-y|^\alpha$$

for all y in an interval around x. The function f is *Lipschitz of order* α *on an interval* if (*) holds for all x and y in the interval.

(a) If f is Lipschitz of order $\alpha > 0$ at x, then f is continuous at x.

Proof: Given $\epsilon > 0$ if we let $\delta = (\epsilon/C)^{1/\alpha}$, that means $|x-y| < \delta$ implies $|f(x) - f(y)| \leq C|x-y|^\alpha < C((\epsilon/C)^{1/\alpha})^\alpha = \epsilon$.

(b) If f is Lipschitz of order $\alpha > 0$ on an interval, then f is uniformly continuous on interval.

Proof: Given $\epsilon > 0$ if we let $\delta = (\epsilon/C)^{1/\alpha}$, that means $|x-y| < \delta$ implies $|f(x) - f(y)| \leq C|x-y|^\alpha < C((\epsilon/C)^{1/\alpha})^\alpha = \epsilon$.

(c) If f is differentiable at x, then f is Lipschitz of order 1 at x. Is the converse true?

Proof: (d) If f is differentiable on $[a, b]$, is f Lipschitz of order 1 on $[a, b]$.

Proof: Since f is differentiable on $[a, b]$ then it is also continuous on $[a, b]$. So we can apply the Mean Value Theorem. This means there exists a $c \in [a, b]$ such that:

$$f(x) - f(y) = f'(c)(x-y)$$

$$|f(x) - f(y)| = |f'(c)||(x-y)|$$

$$|f(x) - f(y)| \leq C|x-y|^1$$

So f is Lipschitz of order 1 on $[a, b]$.

(e) If f is Lipschitz of order $\alpha > 1$ on $[a, b]$, then f is constant on $[a, b]$.

Proof: If f is Lipschitz of order $\alpha > 1$ on $[a, b]$, then:

$$|f(x) - f(y)| \leq C|x - y|^{\alpha}$$

$$\Rightarrow \frac{|f(x) - f(y)|}{|x - y|} \leq C|x - y|^{\alpha - 1}$$

$$\Rightarrow \lim_{y \to x} \frac{|f(x) - f(y)|}{|x - y|} \leq \lim_{y \to x} C|x - y|^{\alpha - 1}$$

$$\Rightarrow |f'(x)| \leq 0$$

This implies that $f'(x) = 0$ for all x. So then we have that f is a constant.

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35. Prove that if

$$\frac{a_0}{1} + \frac{a_1}{2} + \ldots + \frac{a_n}{n+1} = 0,$$

then

$$a_0 + a_1 x + \ldots + a_n x^n = 0$$

for some x in [0,1].

Proof: Let $f(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n$ and consider the antiderivative $F(x) = \int f(x) = a_0 x + \frac{a_1}{2}x^2 + \ldots + \frac{a_{n-1}}{n}x^n + \frac{a_n}{n+1}x^{n+1}$. This function is continuous as well on all of $\mathcal{R}$. So then we have that for some $c \in [0, 1]$

$$F(0) - F(1) = F'(c)(1 - 0)$$

from the Mean Value Theorem on the interval [0,1]. Since $F'(x) = f(x), F(0) = 0$ and conveniently, $F(1) = 0$ we have that

$$0 = f(c)$$

So we have that there exists some $c \in [0, 1]$ such that $f(c) = 0$, which is what we wanted.

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38. (a) Prove that the function $f(x) = x^2 - \cos x$ satisfies $f(x) = 0$ for precisely two numbers x.

Proof: Look at $f(x)$ for $x > 1$, then we have that $x^2 - \cos x > 0 \Rightarrow f(x) > 0$. Similarly, if we look at f when $x < -1$ we get that again $f(x) > 1$. So the only possible place where we could have the two zeroes are in this interval $[-1, 1]$. So now let's look at two more intervals that cut that interval in half, $[-\pi/2, 0)$ and $(0, \pi/2]$. We know that $f(-\pi/2) = \pi^2/4 - 1 > 0$ and $f(0) = 1 > 0$. Since $f(x)$ is obviously continuous, it has the intermediate value property so it will have at least one zero in $[-\pi/2, 0]$. As well we know that $f'(x) = 2x + \sin x$ is negative on that entire interval. So $f(x)$ is strictly decreasing and thus will have at most one zero in $[-\pi/2, 0)$. So f has exactly one zero. The argument follows similarly for the other interval with the only difference being that $f'(x)$ is positive on the interval $(0, \pi/2]$. Since we have from earlier that there are no zeroes outside of the interval [-1,1], we know f has exactly two zeroes.

(b) Prove the same for the function $f(x) = 2x^2 - x \sin x - \cos^2 x$. (Some preliminary estimates will be useful to restrict the possible location of the zeros of f.)

Proof: Using the previous strategy we can restrict all the zeroes of f again to the interval $[-\pi/4, \pi/4]$. We know that $f(-\pi/4) = \pi^2/8 + \pi/4 \sin(-\pi/4) - \cos^2(-\pi/4) = \pi^2/8 - *****\!* > 0, f(0) = -1$, and $f(\pi/4) = \pi^2/8 - \pi/4 \sin(\pi/4) - \cos^2(\pi/4) = \pi^2/8 - ********** > 0$. So by the Intermediate Value Theorem, since f is continuous for all x we have that f must have a zero in each of the intervals $(-\pi/4, 0), (0, \pi/4)$. Then if we look at the derivative:

$$f'(x) = 4x - \sin x - x \cos x + 2 \cos x \sin x$$

$$\Rightarrow f'(x) = 4x + \sin x - x \cos x + \sin 2x$$

So rearrange things so they look like this:

$$f'(x) = x(4 - \cos x) + (\sin 2x - \sin x)$$

So look at the sign of the derivative from $(-\pi/4, 0)$. We have that $x(4 - \cos x)$ is a negative times a positive and for the term $(\sin 2x - \sin x)$, $\sin 2x$ is always greater in absolute value on the interval given so we know that this term will also be negative. So we have that on this entire interval $f'(x) < 0$. So it is strictly decreasing and thus has at most one zero on $(-\pi/4, 0)$. ******************** $(0, \pi/2)$.

---

41. Suppose that f satisfies

$$f''(x) + f'(x)g(x) - f(x) = 0$$

for some function g. Prove that if f is 0 at two points, then f is 0 on the interval between them.

Proof: Let $f(a) = f(b) = 0$ for $a < b$.

---

46. Prove that the conclusion of the Cauchy Mean Value Theorem can be written in the form

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)}$$

under the assumptions that $g(b) \neq g(a)$ and that $f'(x)$ and $g'(x)$ are never simultaneously 0 on (a, b).

Proof: We have from the Cauchy Mean Value Theorem that

$$g'(x)(f(b) - f(a)) = f'(x)(g(b) - g(a))$$

Since $g(b) \neq g(a)$ we can divide both sides by $g(b) - g(a)$ to get:

$$\frac{g'(x)(f(b) - f(a))}{(g(b) - g(a))} = f'(x)$$

We can only divide both sides by $g'(x)$ if $g'(x) \neq 0$. So assume g' does equal zero at some point, say α, then we know that $f'(\alpha) \neq 0$. As well, this would mean that $f'(\alpha)(g(b) - g(a)) = 0 \Rightarrow$

$g(b) - g(a) = 0$. But this contradicts the assumption that $g(b) \neq g(a)$. So we know that $g'(x) \neq 0$ on (a, b) so we can divide both sides by it to get our desired conclusion:

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x)}{g'(x)}$$

---

47. Prove that if f and g are continuous on $[a, b]$ and differentiable on (a, b), and $g'(x) \neq 0$ for x in (a, b), then there is some x in (a, b) with

$$\frac{f'(x)}{g'(x)} = \frac{f(x) - f(a)}{g(b) - g(x)}$$

Hint: Multiply out first, to see what this really says.

Proof:

---

50. Find $f'(0)$ if

$$f(x) = \begin{cases} \frac{g(x)}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

and $g(0) = g'(0) = 0$ and $g''(0) = 17$.

Proof: Start with the definition to get:

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$\Rightarrow f'(0) = \lim_{h \to 0} \frac{f(h) - 0}{h}$$

$$\Rightarrow f'(0) = \lim_{h \to 0} \frac{g(h)/h}{h}$$

$$\Rightarrow f'(0) = \lim_{h \to 0} \frac{g(h)/h - g'(0)}{h}$$

$$\Rightarrow f'(0) = \lim_{h \to 0} \frac{g'(h + 0) - g'(0)}{h}$$

$$\Rightarrow f'(0) = g''(0) = 17$$

---

55. Suppose that f is differentiable in some interval containing a, but that f' is discontinuous at a.

(a) The one-sided limits $\lim_{x \to a^+} f'(x)$ and $\lim_{x \to a^-} f'(x)$ cannot both exist. (This is just a minor variation on Theorem 7.)

Proof:

(b) Neither of these one-sided limits can exist even in the sense of being $+\infty$ or $-\infty$. HInt: Use Darboux's Theorem (Problem 54)

Proof:

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61. Use derivatives to prove that if $n \geq 1$, then

$$(1+x)^n > 1+nx \qquad for-1<x<0and0<x$$

(notice that equality holds for x=0).

Proof: We know that at $x=0$ we have that $(1+0)^n = 1+n(0) \Rightarrow 1=1$. So if we think of the left hand side and the right hand side as two functions that cross at $x=0$, then we want to show that for $x>0$ the left hand side has a derivative that is always greater than the right hand side, and for $-1<x<0$, the derivative of the left hand side is smaller than the derivative of the right hand side. Consider first when $x>0$. Then if we take the derivative of both sides we get:

$$n(1+x)^{n-1} > n \Rightarrow (1+x)^{n-1} > 1$$

Since $x>0$, this inequality is obviously true. For $-1<x<0, (1+x)=y$ where $0<y<1$ so:

$$(1+x)^{n-1} = y^{n-1} < 1$$

which is what we wanted. Now, we have two functions who cross at $x=0$ and they are both monotone increasing functions and $n(x+1)^{n-1} > n$ for $x>0$ so the LHS will be increasing faster than the RHS which implies that $(1+x)^n > 1+nx$ and the same idea applies for when $-1<x<0$.

---

62. Let $f(x) = x^4 \sin^2 1/x$ for $x \neq 0$, and let $f(0)=0$.

(a) Prove that 0 is a local minimum point for f.

Proof: If we look at the derivative and use the Squeeze Theorem it is easy to see that the derivative exists at $x=0$ and that $f'(x)=0$:

$$-1 \leq \sin^2 1/h \leq 1$$

$$-h^3 \leq \frac{h^4 \sin^2 1/h}{h} \leq h^3$$

$$\lim_{h\to 0} -h^3 \leq \lim_{h\to 0} \frac{h^4 \sin^2 1/h}{h} \leq \lim_{h\to 0} h^3$$

$$0 \leq \lim_{h\to 0} \frac{h^4 \sin^2 1/h - 0}{h} \leq 0$$

$$0 \leq f'(0) \leq 0$$

It is only left to show that $x=0$ is a local minimum. It is not hard to see though that the function value is always positive for all $x \neq 0$ because we have $x^4 > 0$ and $[\sin(1/x)]^2 > 0$ as well. In any small neighborhood to the left and the right of $x = 0$ we have that $f(x) > 0$ so this means that $x = 0$ must be a local minimum.

(b) Prove that $f'(0) = f''(0) = 0$.

Proof: From above we know that $f'(0) = 0$. To show that $f''(0) = 0$, do the same thing again:

$$-1 \leq \sin^2 1/h \leq 1$$

$$-4h^3 \leq 4h^3 \sin^2 1/h \leq 4h^3$$

Also we know that:

$$-1 \leq \sin 1/h \leq 1$$

$$-2h^2 \leq 2h^2 \sin^2 1/h \leq 2h^2$$

Now, subtract the second from the first to get:

$$-4h^3 + 2h^2 \leq 4h^3 \sin^2 1/h - 2h^2 \sin 1/h \leq 4h^3 - 2h^2$$

$$-4h^2 + 2h \leq \frac{4h^3 \sin^2 1/h - 2h^2 \sin 1/h}{h} \leq 4h^2 - 2h$$

$$\lim_{h \to 0} -4h^2 + 2h \leq \lim_{h \to 0} \frac{4h^3 \sin^2 1/h - 2h^2 \sin 1/h}{h} \leq \lim_{h \to 0} 4h^2 - 2h$$

$$0 \leq \lim_{h \to 0} \frac{4h^3 \sin^2 1/h - 2h^2 \sin 1/h}{h} \leq 0$$

$$0 \leq \lim_{h \to 0} \frac{f'(0 + h) - f'(0)}{h} \leq 0$$

Which means that $f''(0) = 0$ by the Squeeze Theorem.

This function thus provides another example to show that Theorem 6 cannot be improved. It also illustrates a subtlety about maxima and minima that often goes unnoticed: a function may not be increasing in any interval to the right of a local minimum point, nor decreasing in any interval to the left.

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63. (a) Prove that if $f'(a) > 0$ and f' is continuous at a, then f is increasing in some interval containing a.

Proof: Since $f'(a)$ exists we know that the for some small interval around a, $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists. We are also given that $f(a) > 0 \Rightarrow \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} > 0$. So then we have that for $h > 0$ and in that small interval, $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} > 0$ and $a + h > a$. So it must be the case that $f(a + h) > f(a)$ in the interval $(a, a + h)$. We can do the same thing when $h < 0$ and come to the same conclusion. So if we put these two things together we have an interval $(a - h, a + h)$ where f is increasing.

The next two parts of this problem show that continuity of f' are essential.

(b) If $g(x) = x^2 \sin 1/x$, show that there are numbers x arbitrarily close to 0 with $g'(x) = 1$ and also with $g'(x) = -1$

Proof: We have that $f'(x) = 2x\sin 1/x - \cos 1/x$. Consider the sequence $x_n = \frac{1}{\pi+2n\pi}$ for all $n \in \mathbb{Z}$ then $f'(x_n) = 2x_n \sin 1/x_n - \cos 1/x_n = 0 - (-1) = 1$ and $x_n \to 0$ as $n \to \infty$ while $f(x_n)$ is always 1. As well consider the sequence $y_n = \frac{1}{2n\pi}$ for all $n \in \mathbb{Z}$. We get that $y_n \to 0$ as $n \to \infty$ while $f(y_n) = 2y_n \sin 1/y_n - \cos 1/y_n = 0 - 1 = -1$ for all n.

(c) Suppose $0 < \alpha < 1$. Let $f(x) = \alpha x + x^2 \sin 1/x$ for $x \neq 0$, and let $f(0) = 0$. Show that f is not increasing in any open interval containing 0, by showing that in any interval there are points x with $f'(x) > 0$ and also points x with $f'(x) < 0$.

Proof:

The behavior of f for $\alpha \geq 1$, which is much more difficult to analyze, is discussed in the next problem.

3 Rudin

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1. Let f be defined for all real x, and suppose that

$$|f(x) - f(y)| \leq (x-y)^2$$

for all real x and y. Prove that f is constant.

Proof: Start with what is given:

$$|f(x) - f(y)| \leq (x-y)^2$$

$$|f(x) - f(y)| \leq |x-y|^2$$

$$\frac{|f(x) - f(y)|}{|x-y|} \leq |x-y|$$

Now take the limit of both sides as y approaches x:

$$\lim_{y\to x} \frac{|f(x) - f(y)|}{|x-y|} \leq \lim_{y\to x} |x-y|$$

$$|f'(x)| \leq 0$$

This means that $f'(x)$ is identically zero for all x which is true if and only if $f(x)$ is constant.

---

2. Suppose $f'(x) > 0$ in (a,b). Prove that f is strictly increasing in (a,b), and let g be its inverse function. Prove that g is differentiable, and that

$$g'(f(x)) = \frac{1}{f'(x)} \qquad (a < x < b).$$

Proof: Let $x, y \in (a, b)$ and $x < y$. Since $f'(x) > 0$ we know that f is continuous on $[a, b]$ and differentiable on (a, b), that means we can apply the Mean Value Theorem to get there exists a $c \in (a, b)$:

$$f(y) - f(x) = f'(c)(y - x) > 0$$

$$\Rightarrow f(y) > f(x)$$

So f is an increasing function. Since g is the inverse of f, then $g(y) = x$ for all $f(x) = y$.

$$g'(f(x)) = \lim_{y \to x} \frac{g(f(y)) - g(f(x))}{f(y) - f(x)} = \lim_{y \to x} \frac{y - x}{f(y) - f(x)} = \frac{1}{f'(x)}$$

We only have to worry about this limit not existing when $f'(x) = 0$ but we have already proved that $f'(x) > 0$ for all x. So we are done.

---

3. Suppose g is a real function on R^1, with bounded derivative (say $|g'| \leq M$). Fix $\epsilon > 0$, and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough. (A set of admissible values of ϵ can be determined which depends only on M.)

Proof: We want to show that $f'(x) \geq 0$ for all x because that means that f is monotone increasing which means f is 1-1. To show this consider:

$$f'(x) = 1 + \epsilon g'(x)$$

Since $|g'| \leq M$ that means $-M \leq g'(x) \leq M$ for all x. So then we know that

$$1 - \epsilon M \leq f'(x) \leq 1 + \epsilon M$$

So if we pick $\epsilon = 1/2M$ then we have that

$$1 - (\frac{1}{2M})M \leq f'(x) \leq 1 + (\frac{1}{2M})M$$

$$1 - 1/2 \leq f'(x) \leq 1 + 1/2$$

$$1/2 \leq f'(x) \leq 3/2$$

So $f'(x) > 0$ which is what we needed.

---

4. If

$$C_0 + \frac{C_1}{2} + \ldots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0,$$

where $C_0, ..., C_n$ are real constants, prove that the equation

$$C_0 + C_1 x + \ldots + C_{n-1}x^{n-1} + C_n x^n = 0$$

has at least one real root between 0 and 1.

Proof: Let $f(x) = C_0 + C_1x + \ldots + C_{n-1}x^{n-1} + C_nx^n$ and consider the antiderivative $F(x) = \int f(x) = C_0x + \frac{C_1}{2}x^2 + \ldots + \frac{C_{n-1}}{n}x^n + \frac{C_n}{n+1}x^{n+1}$. This function is continuous as well on all of $\mathcal{R}$. So then we have that for some $c \in [0,1]$

$$F(0) - F(1) = F'(c)(1-0)$$

from the Mean Value Theorem on the interval [0,1]. Since $F'(x) = f(x), F(0) = 0$ and conveniently, $F(1) = 0$ we have that

$$0 = f(c)$$

So we have that there exists some $c \in [0,1]$ such that $f(c) = 0$, which is what we wanted.

---

5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.

Proof: Since f is differentiable, f is continuous for all $x > 0$. So we can use the Mean Value Theorem to say that:

$$g(x) = f(x+1) - f(x) = f'(c)(x+1-x) = f'(c)$$

If we do this for all $x > 0$ we will have an uncountable set $\{c_\alpha\}$. This isn't a sequence because it is in uncountable, but there is a sequence of c_n's of the set that goes to ∞. So this means that $f'(c_n) \to 0$ as $n \to \infty$. So this means that $g(x) \to 0$ as well.

---

6. Suppose
(a) f is continuous for $x \geq 0$,
(b) $f'(x)$ exists for $x > 0$,
(c) $f(0) = 0$,
(d) f' is monotonically increasing.
Put

$$g(x) = \frac{f(x)}{x} \qquad (x > 0)$$

and prove that g is monotonically increasing.

Proof: We want to show that $g'(x) \geq 0$, because this will imply that g is monotonically increasing. So lets look at $g'(x)$:

$$g'(x) = \frac{f'(x)x - f(x)}{x^2} \geq 0$$

Since $x^2 \geq 0$ for all x this reduces to

$$f'(x)x - f(x) \geq 0$$

$$\Rightarrow f'(x)x \geq f(x)$$

$$\Rightarrow f'(x) \geq \frac{f(x)}{x}$$

$$\Rightarrow f'(x) \geq g(x)$$

So it suffices to show that if this is true then g is monotonically increasing. To show this, take any $a > 0$. Then the Mean Value Theorem says that $\exists c \in (0, a)$ such that

$$f(a) - f(0) = f'(c)(a - 0)$$

$$\Rightarrow f'(c) = \frac{f(a)}{a}$$

$$\Rightarrow f'(c) = g(a)$$

Since f' is monotone increasing and $c < a$ we know that

$$f'(a) \geq f'(c) = g(a)$$

And this is true for all $a > 0$. So we have the desired result that $f'(x) \geq g(x)$ for all $x > 0$. So g is monotone increasing as well.

---

7. Suppose $f'(x)$, $g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$

Proof: Consider the definition of the derivative for both functions f and g

$$f'(x) = \lim_{t \to \infty} \frac{f(t) - f(x)}{t - x} \qquad g'(x) = \lim_{t \to \infty} \frac{g(t) - g(x)}{t - x}$$

Now, look at the ratio of the two derivatives

$$\frac{f'(x)}{g'(x)} = \lim_{t \to \infty} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} = \lim_{t \to \infty} \frac{f(t) - f(x)}{g(t) - g(x)} = \lim_{t \to \infty} \frac{f(t)}{g(t)}$$

$$\Rightarrow \frac{f'(x)}{g'(x)} = \lim_{t \to \infty} \frac{f(t)}{g(t)}$$

---

8. Suppose f' is continuous on $[a, b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that

$$|\frac{f(t) - f(x)}{t - x} - f'(x)| < \epsilon$$

whenever $0 < |t - x| < \delta, a \leq x \leq b, a \leq t \leq b$. (This could be expressed by saying that f is uniformly differentiable on $[a, b]$ if f' is continuous on $[a, b]$). Does this hold for vector-valued functions too?

Proof: By the definition of continuity for f', $\forall x, t \in [a, b]$ then

$$|x - t| < \delta \Rightarrow |f'(x) - f'(t)| < \epsilon$$

Now use the Mean Value Theorem to get a c between x and t such that

$$f(t) - f(x) = f'(c)(t - x)$$

$$\Rightarrow f'(c) = \frac{f(t) - f(x)}{t - x}$$

Since c is between x and t we know that if $|x - t| < \delta$ then $|t - c| < \delta$ as well. So we can say that

$$|f'(c) - f'(x)| < \epsilon$$

And if we substitute for $f'(c)$ we get the desired result that

$$\left| \frac{f(t) - f(x)}{t - x} - f'(x) \right| < \epsilon$$

whenever $0 < |t - x| < \delta, a \le x \le b, a \le t \le b$.

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15. Suppose $a \in R^1$, f is a twice-differentiable real function on (a, ∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|, |f'(x)|, |f''(x)|$, respectively on (a, ∞). Prove that

$$M_1^2 \le 4M_0M_2.$$

Does $M_1^2 \le 4M_0M_2$ hold for vector-valued functions too?

Proof: To see that the hint is true we start with the fact that if $h > 0$, Taylor's Theorem says that

$$f'(x) = \frac{1}{2h}[f(x + 2h) - f(x)] - hf''(\xi)$$

for some $\xi \in (x, x + 2h)$. So take the absolute value of both sides to get

$$|f'(x)| = |\frac{1}{2h}[f(x + 2h) - f(x)] - hf''(\xi)|$$

$$\Rightarrow |f'(x)| \le |\frac{1}{2h}f(x + 2h)| + |\frac{1}{2h}f(x)| + |hf''(\xi)|$$

$$\Rightarrow |f'(x)| \le \frac{1}{2h}M_0 + \frac{1}{2h}M_0 + hM_2$$

$$\Rightarrow |f'(x)| \le \frac{2M_0}{2h} + hM_2$$

$$\Rightarrow |f'(x)| \le \frac{M_0}{h} + hM_2$$

$$\Rightarrow 0 \le M_2h - |f'(x)| + \frac{M_0}{h}$$

$$\Rightarrow 0 \le M_2h^2 - |f'(x)|h + M_0$$

So then this function can have at most one real zero since it's graph is always greater than or equal to zero. So this means that $b^2 - 4ac \le 0 \Rightarrow |f'(x)| \le 4M_2M_0 \Rightarrow M_1^2 \le 4M_0M_2$.

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16. Suppose f is twice-differentiable on $(0,\infty)$, f'' is bounded on $(0,\infty)$, and $f(x)\to 0$ as $x\to\infty$. Prove that $f'(x)\to 0$ as $x\to\infty$. Hint: Let $a\to\infty$ in Exercise 15.

Proof: If we follow the hint and let $a\to\infty$ and assume that $|f''(x)|\leq A$, we have from exercise 15 that $M_1^2\leq 4M_0M_2$. There may be no upper bound for f on $(0,\infty)$, but we are given that $f(x)\to 0$ as $x\to\infty$. Putting these two facts together, we get that $4A|f(x)|\to 0$ as $x\to\infty$ and so we have that $|f'(x)|\leq 0$ as $x\to\infty$ which is our desired result.

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17. Suppose f is a real, three times differentiable function on [-1,1], such that

$$f(-1)=0,\qquad f(0)=0,\qquad f(1)=1,\qquad f'(0)=0$$

Prove that $f^{(3)}(x)\geq 3$ for some $x\in(-1,1)$.

Note that equality holds for $\frac{1}{2}(x^3+x^2)$.

Proof: Use Theorem 5.15 to approximate the error in the third derivative of $P(t)$ for $n=3$. Then we get:

$$f(\beta)=P(\beta)+\frac{f^{(3)}(x)}{3!}(\beta-\alpha)^3$$

Following the hint from the book, let's look at $\alpha=0,\beta=1$. Then from Theorem 5.15, there exists some point $s\in(0,1)$ such that

$$f(1)=P(1)+\frac{f^{(3)}(s)}{3!}(1-0)^3$$

$$\Rightarrow 1-P(1)=\frac{f^{(3)}(s)}{3!}$$

As well if we let $\alpha=0,\beta=-1$. Then from Theorem 5.15, there exists some point $t\in(-1,0)$ such that

$$f(-1)=P(-1)+\frac{f^{(3)}(t)}{3!}(-1-0)^3$$

$$\Rightarrow 0-P(-1)=-\frac{f^{(3)}(t)}{3!}$$

If we take the second equation and subtract it from the first we get:

$$[1-P(1)]-[0-P(-1)]=\left[\frac{f^{(3)}(s)}{3!}\right]-\left[-\frac{f^{(3)}(t)}{3!}\right]$$

Now using the fact that $P(1)=P(-1)$ we get that

$$1=\frac{f^{(3)}(s)}{3!}+\frac{f^{(3)}(t)}{3!}$$

$$\Rightarrow 6=f^{(3)}(s)+f^{(3)}(t)$$

So if $f^{(3)}(s) = f^{(3)}(t)$ then we can use the equation above to say that $f^{(3)}(s) + f^{(3)}(s) = 6 \Rightarrow 2f^{(3)}(s) = 6 \Rightarrow f^{(3)}(s) = 3$. Otherwise, assume without loss of generality that $f^{(3)}(s) > f^{(3)}(t)$. Then we have that

$$\begin{gathered} f^{(3)}(s) + f^{(3)}(s) > f^{(3)}(t) + f^{(3)}(s) \\ \Rightarrow 2f^{(3)}(s) > f^{(3)}(t) + f^{(3)}(s) = 6 \\ \Rightarrow f^{(3)}(s) > 3 \end{gathered}$$

So either way, we have a point s such that $f^{(3)}(s) \geq 3$ for some $s \in (-1, 1)$.

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22. Suppose f is a real function on $(-\infty, \infty)$. Call x a *fixed point* of f if $f(x) = x$.

(a) If f is differentiable and $f'(t) \neq 1$ for every real t, prove that f has at most one fixed point.

Proof: Assume to the contrary that f has two fixed points $x_1, x_2 \in \mathbb{R}$. Then $f(x_1) = x_1, f(x_2) = x_2$. Since f is differentiable for every real t then we know that it is continuous. So we can say that for any interval $[x_1, x_2], f$ is continuous and differentiable so we can apply the Mean Value Theorem to say there exists a $c \in [x_1, x_2]$:

$$\begin{gathered} f(x_1) - f(x_2) = f'(c)(x_1 - x_2) \\ \Rightarrow \frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(c) \\ \Rightarrow \frac{x_1 - x_2}{x_1 - x_2} = f'(c) \\ \Rightarrow f'(c) = 1 \end{gathered}$$

This contradicts the fact that $f'(t) \neq 1$ for every real t. So it must be the case that f has at most one fixed point.

(b) Show that the function f defined by

$$f(t) = t + (1 + e^t)^{-1}$$

has no fixed point, although $0 < f'(t) < 1$ for all real t.

Proof: Assume to the contrary that f has a fixed point, call it $x_0 \in \mathbb{R}$. Then

$$\begin{gathered} f(x_0) = x_0 + (1 + e^{x_0})^{-1} = x_0 \\ \Rightarrow (1 + e^{x_0})^{-1} = 0 \\ \Rightarrow \frac{1}{1 + e^{x_0}} = 0 \end{gathered}$$

This is obviously a contradiction, because the left hand side is always positive. So f has no fixed point.

(c) However, if there is a constant $A < 1$ such that $|f'(t)| \leq A$ for all real t, prove that a fixed point x of f exists, and $x = \lim x_n$, where x_1 is an arbitrary real number and

$$x_{n+1} = f(x_n)$$

for $n = 1, 2, 3, ...$

Proof: Since $\lim_{n\to\infty} x_n = x$ we can say that $\lim_{n\to\infty} f(x_n) = f(x)$ as well. Given that $f(x_n) = x_{n+1}$ we can use that to say that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} f(x_n)$. Stringing these three inequalities together we get that $f(x) = x$. And so we have that the limit of this sequence is our fixed point.

(d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2) \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \dots$$

Proof:

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26. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \le A|f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$. Hint: Fix $x_0 \in [a, b]$, let

$$M_0 = \sup |f(x)|, \qquad M_1 = \sup |f'(x)|$$

for $a \le x \le x_0$. For any x,

$$|f(x)| \le M_1(x_0 - a) \le A(x_0 - a)M_0.$$

Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f(x) = 0$ on $[a, x_0]$. Proceed.

Proof: Since f is differentiable on $[a, b]$, we know it is continuous on $[a, b]$. So we can apply the Mean Value Theorem to say that for any interval $[a, x] \subseteq [a, b]$ there exists a $c \in [a, x]$ such that

$$f(x) - f(a) = f'(c)(x - a)$$

$$\Rightarrow |f(x)| = |f'(c)||x - a| \le M_1(x - a) \le M_1(x_0 - a) \le A|f(x)|(x_0 - a) \le A(x_0 - a)M_0$$

$$\Rightarrow |f(x)| \le A(x_0 - a)M_0$$

when $a \le x \le x_0$.

Since $A > 0$ is some fixed number and we have control over x_0, we can pick $x_0 - a$ such that it is strictly less than $1/A$. That way we can say that $A(x_0 - a) < 1$. If this is true then we know that $|f(x)| \le A(x_0 - a)M_0 < M_0$ for all $x \in (a, x_0)$. But this would mean that we have a number $A(x_0 - a)M_0$ strictly smaller than the supremum and still an upper bound for all x which contradicts the fact that M_0 was a sup in the first place. The only way this can be true is if $M_0 = 0$. And if this is true then we would have that $|f(x)| \le 0$ for all $x \in [a, x_0]$ which means f is identically zero on this interval. We can continue this process again on the interval from $[x_0, x_1]$ and get a similar result by letting $x_1 - x_0 = x_0 - a$, then we will have the same situation that $A(x_1 - x_0) < 1 \Rightarrow M_0 = 0$ on $x \in [x_0, x_1]$. Since $x_0 - a$ is some fixed number, eventually we will cover the entire interval $[a, b]$ by the Archimedean Principle.

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27. Let ϕ be a real function defined on a rectangle R in the plane, given by $a \le x \le b, \alpha \le y \le \beta$. A *solution* of the initial-value problem

$$y' = \phi(x, y), \qquad y(a) = c, \qquad (\alpha \le c \le \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c, \alpha \leq f(x) \leq \beta$, and

$$f'(x) = \phi(x, f(x)) \qquad (a \leq x \leq b).$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_1) \in R$.

Proof: Assume we have two solutions f_1, f_2. We will show that they are equal. Let $g(x) = f_2(x) - f_1(x)$. then we have that $g(a) = f_2(a) - f_1(a) = c - c = 0$. As well, we are assuming that

$$|\phi(x, y_2) - \phi(x, y_1)| \leq A|y_2 - y_1|$$

$$\Rightarrow |f_2'(x) - f_1'(x)| \leq A|f_2(x) - f_1(x)|$$

$$\Rightarrow |g'(x)| \leq A|g(x)|$$

So we have all the conditions we need to apply what we proved in Exercise 26. So we have that g is identically zero on $[a, b]$ which means that $f_2(x) - f_1(x) = 0 \Rightarrow f_1(x) = f_2(x)$.