Math 230B Homework 1

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1. Suppose f is a real function defined on R^1 which satisfies

$$\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$$

for every $x \in R^1$. Does this imply that f is continuous?

Proof: No this does not necessarily imply that f is continuous. Consider the "tent" function from class:

$$f(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \end{cases}$$

Let p be an irrational number. Then given $\epsilon > 0$, there exists an $n \in N$ such that $1/n < \epsilon$. Now, since $f(a/n) = 1/n$ for all $n < a$, there are only finitely many numbers of the form a/b where $\gcd(a, b) = 1$ and $a \leq b \leq n$. So then if we let $\delta = \min\{d(a/b, p) : \gcd(a, b) = 1, a \leq b \leq n\}$, we will have that for all x if $|x - p| < \delta$ then $|f(x) - f(p)| = |f(x) - 0| \leq 1/n < \epsilon$. So the $\lim_{x \to a} f(x) = 0$ for all a which means $\lim_{h \to 0}[f(x+h) - f(x-h)] = 0$ for all a, but the function is not continuous at any a equal to a rational number.

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2. If f is continuous mapping of a metric space X into a metric space Y, prove that

$$f(\overline{E}) \subset \overline{f(E)}$$

for every set $E \subset X$. Show, by an example, that $f(\overline{E})$ can be a proper subset of $\overline{f(E)}$.

Proof: Let $x_0 \in \overline{E}$. Then either $x_0 \in E$ or $x_0 \in E'$. So if $x_0 \in E$ then $f(x_0) \in f(E) \subset \overline{f(E)}$. If that isn't the case then $x_0 \in E$ which means that there is a sequence $\{p_n\}$ such that it converges to x_0. That means that $\{f(p_n)\}$ converges to $f(x_0)$. Since every $\{f(p_n)\}$ is in $f(E)$ we know that $f(x_0) \in \overline{f(E)}$

To see that it is a proper subset look at $E = (1, \infty), f(x) = \frac{1}{x}$. Then we have that $\overline{E} = [1, \infty), f(\overline{E}) = (0, 1]$, and $\overline{f(E)} = [0, 1]$. So clearly $f(\overline{E}) \neq \overline{f(E)}$.

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3. Let f be a continuous real function on a metric space X. Let $Z(f)$ (the zero set of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.

Proof: We know that $f(Z(f)) = \{0\}$ because the set $Z(f)$ is defined to be all the values for which $f(x) = 0$ and so the set $\{0\}$ is closed. Since f is continuous we know that $f^{-1}(\{0\})$ is closed. Therefore $Z(f)$ is closed in X.

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4. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for a $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$.

Proof:

Since E is dense in X, for any $p \in X$ we can construct a sequence of points $\{p_n\}$ that converges to p, where each $p_n \in E$. Since f is continuous, we know that $f(p_n) \to f(p)$. Similarly, since g is continuous, we know that $g(p_n) \to g(p)$. Since we have assumed that $f(p) = g(p)$ for all $p \in E$, then we know that $f(p_n) = g(p_n)$ for all n. So this means that $f(p) = \lim_{n\to\infty} f(p_n) = \lim_{n\to\infty} g(p_n) = g(p)$. Since p was chosen arbitrarily from X, we have the desired conclusion that $g(p) = f(p)$ for all $p \in X$.

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5. If f is a real continuous function defined on a closed set $E \subset R^1$, prove that there exist continuous real functions g on R^1 such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called continuous extensions of f from E to R^1.) Show that the result becomes false if the word "closed" is omitted. extend the result to vector-valued functions. Hint: Let the graph of g be a straight line on each of the segments which constitute the complement of E (compare Exercise 29, Chapter 2). The result remains true if R^1 is replaced by any metric space, but the proof is not so simple.

Proof: For E to be a closed set in R^1 it is either a closed interval. So assume $E = [a, b]$ for some $a, b \in R$ then the function g will be

$$g(x) = \begin{cases} f(a) & \text{if } x \leq a \\ f(x) & \text{if } a < x < b \\ f(b) & \text{if } b \leq x \end{cases}$$

Then the function is obviously continuous to the left of a and to the right of b because they are constant functions. As well the $g(x)$ is continuous at every point in the interval (a, b) because $f(x)$ is continuous by assumption. The only points where the function is possibly discontinuous is at a and b. But at a the limit from the right obviously exists because $g(x)$ is continuous and from the left the function value is always $f(a)$, so $\lim_{x\to a^+} = f(a) = \lim_{x\to a^-}$. It follows similarly for $x = b$, so the function is continuous on all of R^1.

For a counterxample, let $f(x) = \frac{1}{x}$ and let $E = (-\infty, 0) \cup (0, \infty)$. So then there is no value at which you can define $f(x)$ to make it continuous at $x = 0$ because the $\lim_{x\to 0+} = \infty$ and so we can't possibly define x to be infinity at zero.

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6. If f is defined on E, the graph of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane.

Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.

Proof: ($\Rightarrow$) Let the f be the map $x \to (x, f(x))$. This map is continuous on E. So then by Theorem 4.14 $f(E) = G$ is compact.

($\Leftarrow$) Assume that the graph is compact. Then the graph is sequentially compact. So that means every sequence $(x_n, f(x_n))$ has a convergent subsequence $(x_{n_j}, f(x_{n_j}))$ that converges to some point $(x_0, f(x_0))$. So each piece converges, which means $\{x_{n_j}\} \to x_0$ and $\{f(x_{n_j})\} \to f(x_0)$. So the $\lim_{n_j \to \infty} f(x_{n_j}) = f(x_0)$ for all $x_0 \in E$, so the function is continuous.

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7. If $E \subset X$ and if f is a function defined on X, the restriction of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$. Define f and g on R^2 by : $f(0,0) = g(0,0) = 0, f(x,y) = \frac{xy^2}{x^2+y^4}, g(x,y) = \frac{xy^2}{x^2+y^6}$ if $(x,y) \neq (0,0)$. Prove that f is bounded on R^2, that g is unbounded in every neighborhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restrictions of both f and g to every straight line in R^2 are continuous!

Proof: At $p = (1,1)$ the function $f(x,y) = 1/2$. Assume that this is the upper bound for $f(x,y)$. Then it must be the case that

$$\begin{aligned} (x - y^2)^2 &\geq 0 \\ \Rightarrow (x^2 - 2xy + y^4) &\geq 0 \\ \Rightarrow x^2 + y^4 &\geq 2xy \\ \Rightarrow \frac{1}{2} &\geq \frac{2xy}{x^2 + y^4} \end{aligned}$$

It follows similarly that if we start with $-(x - y^2)^2 \leq 0$ we will get the other half of the desired equality below

$$\Rightarrow |\frac{xy^2}{x^2 + y^4}| \leq \frac{1}{2}$$

So then $f(x,y)$ is bounded.

To show that f is discontinuous at $(0,0)$, we take the limit along any line, and then we take the limit along the line $y = \sqrt{x}$. Along the first line we get that the limit is in fact zero as the problem states. Along the curve $y = \sqrt{x}$ the limit is

$$\lim \frac{xy^2}{x^2 + y^4} = \lim \frac{x(\sqrt{x})^2}{x^2 + (\sqrt{x})^4} = \lim \frac{x^2}{x^2 + x^2} = \frac{1}{2}.$$

Finally, show that $g(x)$ is unbounded. Then let $x_n = 1/n^3, y_n = 1/n$, so then

$$g(x_n, y_n) = \frac{(1/n^3)(1/n)^2}{(1/n^3) + (1/n)^6}$$

$$\Rightarrow g(x_n, y_n) = \frac{1/n^5}{1/n^6 + 1/n^6}$$

$$\Rightarrow g(x_n, y_n) = \frac{1/n^5}{2/n^6}$$

$$\Rightarrow g(x_n, y_n) = \frac{n}{2}$$

So the $\lim g(x_n, y_n) \to \infty$ as $n \to \infty$.

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8. Let f be a real uniformly continuous function on the bounded set E in R^1. Prove that f is bounded on E.

Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

Proof: Assume f is not bounded. Then for all $M > 0$, there exists and $x \in E$ such that $|f(x)| > M$. Since f is uniformly continuous we know that for $\epsilon > 0$ there exists a $\delta > 0$ such that $|x - y| < \delta$ means $|f(x) - f(y)| < \epsilon$.

If boundedness is omitted, then we can let $f(x) = \sqrt{x}$ and $E = [0, \infty)$. Then we have that f is uniformly continuous, but it is not bounded because $\lim_{x \to \infty} f(x) = \infty$.

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10. Complete the details of the following alternative proof of Theorem 4.19: If f is not uniformly continuous, then for some $\epsilon > 0$ there are sequences $\{p_n\}, \{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \epsilon$. Use Theorem 2.37 to obtain a contradiction.

Proof:

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11. Suppose f is uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X. Use this result to give an alternative proof of the theorem stated in Exercise 13.

Proof: Since $\{x_n\}$ is Cauchy, we know that for all $\epsilon > 0$ there exists an N such that for all $m, n > N, d_X(f(x_m), f(x_n)) < \epsilon$. As well, since f is uniformly continuous we know that for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d_X(p, q) < \delta$ then $d_Y(f(p), f(q)) < \epsilon$ and this δ works for all x. So if we fix an $\epsilon > 0$ then we can find a δ using uniform continuity that works for all x such that there exists an N_δ such that for all $m, n > N_\delta, d_X(x_m, x_n) < \delta \Rightarrow d_Y(f(x_m) - f(x_n)) < \epsilon$ by uniform continuity again. So $\{f(x_n)\}$ is a Cauchy sequence in Y for every Cauchy sequence $\{x_n\}$ in X.

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13. Let E be a dense subset of a metric space X, and let f be a uniformly continuous real function defined on E. Prove that f has a continuous extension from E to X. (Uniqueness follows from exercise 4). Hint: For each $p \in X$ and each positive integer n, let $V_n(p)$ be the set of all $q \in E$ with $d(p,q) < 1/n$. Use Exercise 9 to show that the intersection of the closures of the sets $f(V_1(p)), f(V_2(p))$, ..., consists of a single point, say $g(p)$, of R^1. Prove that the function g so defined on X is the desired extension of f.

Could the range space of R^1 be replaced by R^k? By any compact metric space? By any complete metric space? By any metric space?

Proof:

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14. Let $I = [0,1]$ be the closed unit interval. Suppose f is a continuous mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.

Proof: Construct $h(x) = f(x) - x$. Then we are looking for when $h(x) = 0$. So if $f(0) = 0$ or $f(1) = 1$, then we are done. So assume neither case is true. Then we have that $f(0) > 0$ because f must take on a value in the interval [0,1], and $f(1) < 1$ for the same reason. These two statements imply that $h(0) > 0$ and $h(1) < 0$. So then by the Intermediate Value Theorem, we can say that $h(x) = 0$ for some $x \in [0,1]$ which means that $f(x) = x$ for some $x \in [0,1]$.

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15. Call a mapping of X into Y open if $f(V)$ is an open set in Y whenever V is open in X.

Prove that every continuous open mapping of R^1 into R^1 is monotonic.

Proof:

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18. Every rational x can be written in the form $x = m/n$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take $n = 1$. Consider the function f defined on R^1 by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Proof: Let p be an irrational number. Then given $\epsilon > 0$, there exists an $n \in N$ such that $1/n < \epsilon$. Now, since $f(a/n) = 1/n$ for all $n < a$, there are only finitely many numbers of the form a/b where $\gcd(a,b) = 1$ and $a \leq b \leq n$. So then if we let $\delta = \min\{d(a/b,p) : \gcd(a,b) = 1, a \leq b \leq n\}$, we will have that for all x if $|x - p| < \delta$ then $|f(x) - f(p)| = |f(x) - 0| \leq 1/n < \epsilon$. So the $\lim_{x \to a} f(x) = 0$ for all a which means that f is continuous at every irrational number and discontinuous at every rational number.

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19. Suppose f is a real function with domain R^1 which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b.

Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed. Prove that f is continuous. Hint: If $x \to x_0$ but $f(x_n) > r > f(x_0)$ for some r and all n, then $f(t_n) = r$ for some t_n between x_0 and x_n; thus $t_n \to x_0$. Find a contradiction.

Proof: Let $A = \{x | f(x) = r\}$ and we are given that A is closed. Then we can construct a sequence $\{x_n\}$ that approaches x_0 and since f has the IVP we can say that there exists a $x_n < t_n < x_0$ such that that $f(x_n) < r < f(x_0)$ where r is rational and $f(t_n) = r$. So then $t_n \to x_0$ as well and so it is a limit point of A. Since A is closed, x_0 is in A which means $f(x_0) = r$ which contradicts the fact that $f(x_n) < r$ for all n. So $\lim f(x_n) = f(x_0)$ for all x_0 so f is continuous.

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20. If E is nonempty subset of a metric space X, define the distance from $x \in X$ to E by

$$\rho_E(x) = inf_{z \in E} d(x, z).$$

(a) Prove that $\rho_E(x) = 0$ if and only if $x \in \overline{E}$.

Proof: ($\Leftarrow$) Assume to the contrary that $x \in \overline{E}$ but $\rho_E(x) \neq 0$. If $x \in E$ then we immediately come to a contradiction because $\rho_E(x) = 0$. If $x \in E'$ then there exists a sequence of points $\{x_n\}$ in E the converge to x. So given any $\epsilon > 0$ there is a point x_i such that $d(x_i, x) < \epsilon$. Since ϵ can be chosen arbitrarily small, $\rho_E(x) = 0$. So we get a contradiction either way. So it must be the case that $\rho_E(x) = 0$.

($\Rightarrow$) Suppose $x \notin \overline{E}$ but that $\rho_E(x) = 0$ then $x \in \overline{E}^c$ open, so there exists an $\epsilon > 0$ such that $B(x, \epsilon) \subset \overline{E}^c$, so $\inf d(x, z) \geq \epsilon > 0$ for $z \in E$ which means $\rho_E(x) \neq 0$. This is a contradiction. So $x \in \overline{E}$

(b)Prove that ρ_E is a uniformly continuous function on X, by showing that

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y)$$

for all $x \in X, y \in X$. Hint: $\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z)$, so that $\rho_E(x) \leq d(x, y) + \rho_E(y)$.

Proof: Well if we just continue where the hint leaves off we get $\rho_E(x) - \rho_E(y) \leq d(x, y)$. Similary, $\rho_E(y) \leq d(y, z) \leq d(y, x) + d(x, z)$, so that

$$\rho_E(y) \leq d(x, y) + \rho_E(x)$$

$$\Rightarrow \rho_E(y) - \rho_E(x) \leq d(x, y)$$

$$\Rightarrow d(x, y) \leq \rho_E(x) - \rho_E(y).$$

Combining this with the other inequality above we get $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$.

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21. Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K, q \in F$. Hint: ρ_F is a continuous positive function on K.

Show that the conclusion may fail for two disjoint closed sets if neither is compact.

Proof: Since ρ_F is a continuous positive function from Exercise 20, we know that $\rho_F(K)$ is also a compact set which means $\rho_F(K)$ is closed. If we assume $0 \in \rho_F(K)$ then we know there is a point $x \in K$ such that $x \in \overline{F} = F$ from Exercise 20. This means that $x \in F \cap K$ which is a contradiction to the fact that they are disjoint. So there must exist a value $c > 0$ such that $\rho_F(p) \geq c$ for all $p \in K$. Since we do this for all the points in K and the function ρ_F ranges over points in F we know that this will be the smallest possible distance between any pair of points p, q. If we let $\delta = c$ then we will have that $d(p, q) > \delta$ for all p, q.

Without compactness the proof falls apart at the beginning because when E is closed, and f is continuous $f(E)$ is not necessarily closed.

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10.

(a) Prove that if f is continuous at a, then so is $|f|$.

Proof: If f is continuous at a then for all $\epsilon > 0$ there exists a $\delta > 0$ such that if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$. We also know that $||f(x)| - |g(x)|| < |f(x) - f(a)| < \epsilon$. So we are done.

(b) Prove that every continuous f can be written $f = E + O$, where E is even and continuous and O is odd and continuous

Proof: Let $f(x)$ be a continuous function. Then we can construct $E(x) = \frac{f(x)+f(-x)}{2}$ and $O(x) = \frac{f(x)-f(-x)}{2}$ from $f(x)$. Then $E(-x) = \frac{f(-x)+f(x)}{2} = E(x)$ and $O(x) = \frac{f(-x)-f(x)}{2} = -O(x)$, so $E(x)$ is even and $O(x)$ is odd. They are also both obviously continuous because f itself is continuous and they are just linear combinations of f. Finally, $O(x)+E(x) = \frac{f(x)-f(-x)}{2} + \frac{f(x)+f(-x)}{2} = \frac{2f(x)}{2} = f(x)$

(c) Prove that if f and g are continuous, then so are max(f,g) and min(f,g).

Proof: Well it is easy to verify that

$$\max(f, g) = \frac{f(x) + g(x) + |f(x) - g(x)|}{2}$$
$$\min(f, g) = \frac{f(x) + g(x) - |f(x) - g(x)|}{2}$$

From part (a) we have that if f is continuous then $|f|$ is continuous and since the sum of continuous functions are continuous then max(f,g) and min(f,g) are continuous.

(d) Prove that every continuous f can be written $f = g - h$, where g and h are nonnegative and continuous.

Proof: So let $f(x) = g(x) - h(x) = |\max(f, 0)| - |\min(f, 0)|$. From part (a) and (c) we know that this function will be continuous. So then we have that if $f(x) \geq 0$, $f(x) = |f(x)| - 0 = f(x)$ and if $f(x) < 0$ then we get that $f(x) = 0 - |f(x)| = 0 - (-f(x)) = f(x)$.

13.

(a) Prove that if f is continuous on $[a, b]$, then there is a function g which is continuous on $\mathbf{R}$, and which satisfies $g(x) = f(x)$ for all $x \in [a, b]$.

Proof: If we just define the function to be

$$g(x) = \begin{cases} f(a) & \text{if } x \leq a \\ f(x) & \text{if } a < x < b \\ f(b) & \text{if } b \leq x \end{cases}$$

Then the function is obviously continuous to the left of a and to the right of b because they are constant functions. As well the $g(x)$ is continuous at every point in the interval (a, b) because $f(x)$ is continuous by assumption. The only points where the function is possibly discontinuous is at a and b. But at a the limit from the right obviously exists because $g(x)$ is continuous and from the left the function value is always $f(a)$, so $\lim_{x \to a^+} = f(a) = \lim_{x \to a^-}$. It follows similarly for $x = b$, so the function is continuous on all of R^1.

(b) Give an example to show that this assertion is false if $[a, b]$ is replaced by (a, b).

Proof: Let $f(x) = 1/x$ and let $(a, b) = (0, 1)$. Then no matter what $f(0)$ is defined to be, $\lim_{x \to 0^+} = \infty$, which means that if we try to define a function $g(x)$ to be continuous on all of $\mathbf{R}$, it will always be discontinuous at $x = 0$.

14.

(a) Suppose that g and h are continuous at a, and that $g(a) = h(a)$. Define $f(x)$ to be $g(x)$ if $x \geq a$ and $h(x)$ if $x \leq a$. Prove that f is continuous at a.

Proof: We know that $\lim_{x \to a^-} h(x) = h(a) = \lim_{x \to a^+} h(x)$ and $\lim_{x \to a^-} g(x) = g(a) = \lim_{x \to a^+} g(x)$. As well we have that $g(a) = h(a)$. So we know that $\lim_{x \to a^-} f(x) = \lim_{x \to a^-} h(x) = h(a) = g(a) = \lim_{x \to a^+} g(x) = \lim_{x \to a^+} f(x)$. So the limit from the left and the right equal the value of the function at a, so f is continuous is a.

(b) Suppose g is continuous on $[a, b]$ and h is continuous on $[b, c]$ and $g(b) = h(b)$. Let $f(x)$ be $g(x)$ for x in $[a, b]$ and $h(x)$ for x in $[b, c]$. Show that f is continuous on $[a, c]$.

Proof: Well it is assumed that g, f are continuous on their respective intervals so the f will be continuous at all the points in the $[a, b)$ and $(b, c]$ for sure. The only point that may have a problem is $x = b$ but we can again see that $\lim_{x \to b^-} f(x) = \lim_{x \to b^-} g(x) = g(b) = h(b) = \lim_{x \to b^+} h(x) = \lim_{x \to b^+} f(x)$. So the function is continuous at $x = b$.

15.

(a) Prove the following version of Theorem 3 for "right-hand continuity": Suppose that $\lim_{x \to a^+} f(x) = f(a)$ and $f(a) > 0$. then there is a number $\delta > 0$ such that $f(x) > 0$ for all x satisfying $0 \leq x - a < \delta$. Similarly, if $f(a) < 0$, then there is a number $\delta > 0$ such that $f(x) < 0$ for all x satisfying $0 \leq x - a < \delta$.

Proof: Suppose the $\lim_{x\to a+} f(x) = f(a)$. Then let $\epsilon = f(a) > 0$. Since $|f(x) - f(a)| < \epsilon$ that means $|f(x) - f(a)| < f(a) for 0 < x - a < \delta \Rightarrow -f(a) < f(x) - f(a) < f(a) \Rightarrow 0 < f(x) < 2f(a)$. So there exists a delta such that $f(x) > 0$ for all $0 \leq x - a < \delta$

Similarly, we know that $\lim_{x\to a+} f(x) = f(a)$ and $f(a) < 0$. Then let $\epsilon = -f(a) > 0$. THen we get that $|f(x) - f(a)| < -f(a) \Rightarrow f(a) < f(x) - f(a) < -f(a) \Rightarrow 2f(a) < f(x) < 0$. So there exists a $\delta > 0$ such that $f(x) < 0$ for all x satisfying $0 \leq x - a < \delta$

(b) Prove a version of Theorem 3 when $\lim_{x\to b^-} f(x) = f(b)$.

Proof: This is exactly the same as the part (a). We just let $\epsilon = f(b)$ and $\epsilon = -f(b)$ and we get the same results.

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16. If $\lim_{x\to a} f(x) = f(a)$ exists, but is $\neq f(a)$, then f is said to have a **removable discontinuity** at a.

(a) If $f(x) = \sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 1$, does f have a removable discontinuity at 0? What if $f(x) = x\sin\frac{1}{x}$ for $x \neq 0$, and $f(0) = 1$?

Proof: If $f(x) = \sin\frac{1}{x}$, then the $\lim_{x\to 0} f(x)$ does not exist, so there is no removable discontinuity. But if $f(x) = x\sin\frac{1}{x}$, then the $\lim_{x\to a} f(x) = 0$ while $f(0) = 1$ so that means there is a removable discontinuity at 0.

(b) Suppose f has a removable discontinuity at a. let $g(x) = f(x)$ for $x \neq a$, and let $g(a) = \lim_{x\to a} f(x)$. Prove that g is continuous at a.

Proof: We know that $\lim_{x\to a} g(x) = \lim_{x\to a} f(x) = g(a)$, which is the condition for continuity for $g(x)$ at a.

(c) Let $f(x) = 0$ if x is irrational, and let $f(p/q) = 1/q$ if p/q is in lowest terms. What is the function g defined by $g(x) = \lim_{y\to x} f(y)$?

Proof: $g(x) = 0$.

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4.

(a) If $n - k$ is even, and ≥ 0, find a polynomial function of degree n with exactly k roots.

Proof: Well if the function has k roots and $n - k$ is even, then there are $n - k$ complex roots. If it is required that every root be distinct then we can create the function $(x^2+1)(x^2+2)\ldots(x^2+\frac{n-k}{2})(x-1)(x-2)\ldots(x-k)$, and this obviously has k real roots and the other roots will be complex.

(b) A root a of the polynomial function f is said to have multiplicity m if $f(x) = (x-a)^m g(x)$, where g is a polynomial function that does not have a as a root. Let f be a polynomial function of degree n. Suppose that f has k roots, counting multiplicities, i.e. suppose that k is the sum of the multiplicities of all the roots. Show that $n - k$ is even.

Proof: Let m_i be the multiplicity of the b_ith root. Then $m_1+m_2+....+m_j = k$ where the function has j distinct real roots and $f(x) = (x-b_1)^{m_1}(x-b_2)^{m_2}\ldots(x-b_n)^{m_j}$. Then $\deg[f(x)] = k + \deg[g(x)]$

and since $g(x)$ doesn't have any real roots, the $\deg[g(x)]$ is even by Theorem 9 from Spivak. That means $n - k$ is even as well.

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5. Suppose that f is continuous on $[a, b]$ and that $f(x)$ is always rational. What can be said about f?

Proof: Assume that $f(x)$ is continuous and takes on two different rational values, then by the Intermediate Value Theorem it must take on all values between those two rational numbers which would obviously include irrational numbers. So we know that $f(x)$ must be constant if it is continuous and always rational.

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6. Suppose that f is a continuous function on $[-1, 1]$ such that $x^2 + (f(x))^2 = 1$ for all x. Show that either $f(x) = \sqrt{1 - x^2}$ for all x, or else $f(x) = -\sqrt{1 - x^2}$ for all x.

Proof: Solve for $f(x)$:

$$x^2 + (f(x))^2 = 1$$

$$\Rightarrow (f(x))^2 = 1 - x^2$$

$$\Rightarrow f(x) = \pm\sqrt{1 - x^2}$$

Which gives us the desired result.

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7. How many continuous functions f are there which satisfy $(f(x))^2 = x^2$ for all x?

Proof: If we take the square root of both sides the question is then what continuous functions satisfy $|f(x)| = |x|$. There are four continuous functions: $f(x) = x$, $f(x) = -x$, $f(x) = |x|$, and $f(x) = -|x|$

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10. Suppose f and g are continuous on $[a, b]$ and that $f(a) < g(a)$, but $f(b) > g(b)$. Prove that $f(x) = g(x)$ for some x in $[a, b]$.

Proof: Consider the function $h(x) = f(x) - g(x)$. This function will be continuous because it is the difference of two continuous functions and it will have a zero whenever $f(x) = g(x)$. Since $h(a) = f(a) - g(a) < 0$ and $h(b) = f(b) - g(b) > 0$ we can say that there exists some $x \in [a, b]$ such that $h(x) = 0 \Rightarrow f(x) = g(x)$.

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11. Suppose that f is continuous function on [0,1] and that $f(x)$ is in [0,1] for each x. Prove that $f(x) = x$ for some number x.

Proof: Well again construct $h(x) = f(x) - x$. Then we are looking for when $h(x) = 0$. So if $f(0) = 0$ or $f(1) = 1$, then we are done. So assume neither case is true. Then we have that $f(0) > 0$ because f must take on a value in the interval [0,1], and $f(1) < 1$ for the same reason. These two statements imply that $h(0) > 0$ and $h(1) < 0$. So then we can again apply the same reasoning as exercise 10 to say that $h(x) = 0$ for some $x \in [0, 1]$ which means that $f(x) = x$ for some $x \in [0, 1]$.

12.

(a) Problem 11 shows that f intersects the diagonal of the square made up of [0,1] x [0,1]. Show that f must also intersect the other diagonal.

Proof: The equation of the line for the other diagonal is $f(x) = -x + 1$. So then $f(0) < 1$ and $f(1) > 0$ because otherwise we would be done. So let $h(x) = f(x) + x - 1$ and we want to find out if $h(x)$ has a zero in the interval [0,1]. So then we know that $h(0) < 0$ and $h(1) > 1$. So then we can again apply the same reasoning as exercise 10 to say that $h(x) = 0$ for some $x \in [0, 1]$ which means that $f(x) = -x + 1$ for some $x \in [0, 1]$ which means it intersects the other diagonal.

(b) Prove the following more general fact: If g is continuous on [0,1] and $g(0) = 0, g(1) = 1$ or $g(0) = 1, g(1) = 0$, then $f(x) = g(x)$ for some x.

Proof: Again let $h(x) = f(x) - g(x)$ then we are looking for a zero for h in the interval [0,1]. Again we have that either $h(0) < 0 and h(1) > 1$ or vice versa $h(0) > 0 and h(1) < 1$. So then we can again apply the same reasoning as exercise 10 to say that $h(x) = 0$ for some $x \in [0, 1]$ which means that $f(x) = g(x)$ for some $x \in [0, 1]$

13.

(a) Let $f(x) = \sin 1/x$ for $x \neq 0$ and let $f(0) = 0$. Is f continuous on [-1,1]? Show that f satisfies the conclusion of the Intermediate Value Theorem on [-1,1]; in other words, if f takes on two values on [-1,1], it also takes on every value in between.

Proof: No $f(x)$ is not continuous even if $f(0) = 0$. Let $x = \pi/2$ and let $y = \pi/2$. Then $f(x) = \sin(\pi/2) = 1$ and $f(x) = \sin(-\pi/2) = -1$. So f is continuous on the interval from $[1/4\pi, 1/2\pi]$ and it goes through an entire cycle, so f takes on all the values between -1 and 1 on just the interval $[1/4\pi, 1/2\pi]$ and so it satisfies the Intermediate Value Property.

(b) Suppose that f satisfies the conclusion of the Intermediate Value Theorem, and that f takes on each value only once. Prove that f is continuous.

Proof: Well if f takes on each value once, then the function is one to one. So this means that f is monotone. Since the f has the IVP we can say that it has no jump discontinuities. Taking these two facts together means f is continuous.

(c) Generalize to the case where f takes on each value only finitely many times.

Proof:

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16. Let f be a polynomial function. Prove that there is some number y such that $|f(y)| \leq |f(x)|$ for all x.

Proof: If the $f(x)$ crosses the axis at lets say $y = a$. Then $|f(y)| = 0 \leq |f(x)|$ for all x. If the function doesn't cross the x-axis then the function is always positive or always negative. If the function is always positive and the function is degree n lets say, then there are $\frac{n-1}{2}$ minimums because the function is continuous. Let the set $\{y_1, y_2, ..., y_{\frac{n-1}{2}}\}$ be the points at which the function achieves its minimums. Then $y = \min\{y_1, y_2, ..., y_{\frac{n-1}{2}}\}$ will be the value we need such that $|f(y)| \leq |f(x)|$ for all x. We can use the same argument for a function that is always negative because when we look at $|f(x)|$ the function will become always positive and all the maximums of the function will become minimums.

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17. Suppose that f is a continuous function with $f(x) > 0$ for all x, and $\lim_{x\to\infty} f(x) = 0 = \lim_{x\to-\infty} f(x)$. Prove that there is some number y such that $f(y) \geq f(x)$ for all x.

Proof: Pick any number $M > 0$. Then there are only finitely many local maximums greater than any given M, and call the set of all values where these local maximums are achieved $A = \{x_1, x_2, ..., x_n\}$. Since this set is finite, we can take the maximum to get $y = \max A$. Then $f(y) \geq f(x)$ for all x.
