Math 230a Homework 4

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1. Prove that the convergence of $\{s_n\}$ implies the convergence of $\{|s_n|\}$. Is the converse true?

Proof: Suppose $\{s_n\}$ converges to s. So we know that $|s_n - s| < \epsilon$. We earlier proved from the triangle inequality that $||a| - |b|| \leq |a - b|$. If we replace a and b with s_n and s we get $||s_n| - |s|| \leq |s_n - s|$. So we have that $||s_n| - |s|| \leq |s_n - s| < \epsilon$. So we have that $\{|s_n|\}$ converges to $|s|$. But since it is an inequality, we do not necessarily have that if $\{|s_n|\}$ converges then $\{s_n\}$ converges.

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2. Calculate $\lim_{n\to\infty}(\sqrt{n^2+n} - n)$.

Proof: If we multiply by the conjugate we get:

$$\lim_{n\to\infty}(\sqrt{n^2+n} - n)\frac{(\sqrt{n^2+n}+n)}{(\sqrt{n^2+n}+n)}$$

$$= \lim_{n\to\infty}\frac{n^2+n-n^2}{(\sqrt{n^2+n}+n)}$$

$$= \lim_{n\to\infty}\frac{n}{(\sqrt{n^2+n}+n)}$$

$$= \lim_{n\to\infty}\frac{\frac{n}{n}}{(\sqrt{\frac{n^2}{n^2}+\frac{n}{n^2}}+\frac{n}{n})}$$

$$= \lim_{n\to\infty}\frac{1}{(\sqrt{1+\frac{1}{n}}+1)}$$

$$= \frac{1}{2}$$

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3. If $s_1 = \sqrt{2}$, and

$$s_{n+1} = \sqrt{2 + \sqrt{s_n}} \qquad (n = 1, 2, 3, ...),$$

prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, ...$.

Proof: We can see that $0 < \sqrt{2} < 2 \Rightarrow 2 < 2 + \sqrt{2} < 4 \Rightarrow \sqrt{2} < \sqrt{2 + \sqrt{2}} < 2$. Applying this rule again we get that $\sqrt{2 + \sqrt{2}} < \sqrt{2 + \sqrt{2 + \sqrt{2}}} < 2$. So we can guess that $s_n < \sqrt{2 + s_n} < 2$ $\forall n \in \mathbf{N}$. To prove this use induction. We have the case for $n = 1 : \sqrt{2} < \sqrt{2 + \sqrt{2}} < 2$. So now, assume its true for n and show its true for $n + 1$. So $s_{n+1} = \sqrt{2 + s_n} < \sqrt{2 + \sqrt{2 + s_n}} < 2 \Rightarrow s_{n+1} < \sqrt{2 + s_{n+1}} = s_{n+2} < 2$. So we have a function that is always increasing and bounded above by 2 so the sequence converges by the Monotone Convergence Principle.

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4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by

$$s_1 = 0; \qquad s_{2m} = \frac{s_{2m-1}}{2}; \qquad s_{2m+1} = \frac{1}{2} + s_{2m}.$$

Proof: If we look at the first few terms of the sequence we have that:

$$s_{2m} = 0, \frac{1}{4}, \frac{3}{8}, \frac{7}{16}, ...$$

$$s_{2m+1} = 0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, ...$$

By inspection we can see that the $\liminf_{n\to\infty} s_n = \liminf_{n\to\infty} s_{2n} = \frac{1}{2}$ and $\limsup_{n\to\infty} s_n = \limsup_{n\to\infty} s_{2n+1} = 1$.

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5. For any two real sequences $\{a_n\}, \{b_n\}$, prove that

$$\limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n,$$

provided the sum on the right is not of the form $\infty - \infty$.

Proof: We know that

$$\sup\{a_n + b_n, a_{n+1} + b_{n+1}, ...\} \le \sup\{a_n, a_{n+1}, ...\} + \sup\{b_n, b_{n+1}, ...\}$$

$$\Rightarrow \lim_{n\to\infty} \sup\{a_n + b_n, a_{n+1} + b_{n+1}, ...\} \le \lim_{n\to\infty} (\sup\{a_n, a_{n+1}, ...\} + \sup\{b_n, b_{n+1}, ...\})$$

$$\Rightarrow \limsup_{n\to\infty}(a_n + b_n) \le \lim_{n\to\infty} (\sup\{a_n\} + \sup\{b_n\})$$

$$\Rightarrow \limsup_{n\to\infty}(a_n + b_n) \le \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

6. Investigate the behavior (convergence or divergence) of $\sum a_n$ if

(a) $a_n = \sqrt{n+1} - \sqrt{n}$;

Proof: $a_n = (\sqrt{n+1} - \sqrt{n})\frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})} = \frac{n+1-n}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{n+1}+\sqrt{n}} \geq \frac{1}{2n}$. So $\sum a_n$ diverges by the p-series test.

(b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$;

Proof: $a_n = \frac{(\sqrt{n+1}-\sqrt{n})}{n}\frac{(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n+1}+\sqrt{n})} = \frac{n+1-n}{n(\sqrt{n+1}+\sqrt{n})} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} \leq \frac{1}{n\sqrt{n}}$. So $\sum a_n$ converges by the p-series test with p = 3/2.

(c) $a_n = (\sqrt[n]{n} - 1)^n$;

Proof: Use the root test:

$$\sqrt[n]{a_n} = \sqrt[n]{(\sqrt[n]{n}-1)^n} = \sqrt[n]{n} - 1$$

$$\Rightarrow \limsup_{n\to\infty} \sqrt[n]{a_n} = \limsup_{n\to\infty} \sqrt[n]{n} - 1 = 1 - 1 = 0$$

So $\limsup_{n\to\infty} \sqrt[n]{a_n} = 0 < 1 \Rightarrow \sum a_n$ converges.

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7. Prove that the convergence of $\sum a_n$ implies the convergence of

$$\sum \frac{\sqrt{a_n}}{n},$$

if $a_n \geq 0$.

Proof: By the Schwarz Inequality:

$$\left(\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n}\right)^2 \leq \sum_{n=1}^{N} \sqrt{a_n}^2 \sum_{n=1}^{N} \frac{1}{n^2}$$

$$\Rightarrow \sum_{n=1}^{N} \frac{\sqrt{a_n}}{n} \leq \left(\sum_{n=1}^{N} a_n\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} \frac{1}{n^2}\right)^{\frac{1}{2}}$$

and this is true for all $N \in \mathbf{N}$. Since $\sum \frac{1}{n^2}$ converges by the p-series test and we are given that $\sum a_n$ converges, we can say:

$$\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n} \leq \left(\sum_{n=1}^{N} a_n\right)^{\frac{1}{2}} \left(\sum_{n=1}^{N} \frac{1}{n^2}\right)^{\frac{1}{2}} \leq \sum_{n=1}^{\infty} a_n \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Since both series are convergent we can say the product of any finite sum of terms is less than the infinite sum. So we have that $\sum \frac{\sqrt{a_n}}{n}$ is always bounded above by $\sum a_n \sum \frac{1}{n^2} < \infty$. Since $a_n \geq 0$ for all n, we know that $\frac{\sqrt{a_n}}{n} \geq 0$. Since the terms are all positive we know the sequence of partial sums $\{s_n\}$ is bounded above and monotonic increasing, so we have that the sum converges.

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8. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n b_n$ converges.

Proof: Consider the case where $\{b_n\}$ is monotone increasing and bounded above by say B, then $b_n \leq B$ for all n. So then $\sum a_n b_n \leq \sum a_n B = B \sum a_n = AB$ where A is the infinite sum of the a_n's. The other case is if $\{b_n\}$ are decreasing and bounded below. Then we can not use the comparison test with the lower bound of $\{b_n\}$. But we do know that $b_n < b_1$ for all $n > 2$ because the sequence is monotone decreasing. So we have that $\sum a_n b_n \leq \sum a_n b_1 = b_1 \sum a_n = b_1 A$. So in either case $\sum a_n b_n$ converges.

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9. Find the radius of convergence of each of the following series:

(a) $\sum n^3 z^n$

Proof: Use the root test:

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|n^3|} = \limsup_{n\to\infty} (n^{1/n})^3 = 1 \Rightarrow R = \frac{1}{\alpha} = 1.$$

(b) $\sum \frac{2^n}{n!} z^n$

Proof: The $\sqrt[n]{n!}$ is harder to evaluate so use the ratio test:

$$\alpha = \limsup_{n\to\infty} \frac{\frac{|2^{n+1}|}{|(n+1)!|}}{\frac{|2^n|}{|n!|}} = \limsup_{n\to\infty} \frac{|2^{n+1}|}{|(n+1)!|}\frac{|n!|}{|2^n|} = \limsup_{n\to\infty} \frac{2}{n+1} = 0 \Rightarrow R = \frac{1}{\alpha} = \infty.$$

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10. Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.

Proof: Since $\sum a_n z^n$ converges we know that $\lim_{n\to\infty} a_n z^n = 0$. Since the limit exists we can say that $\lim_{n\to\infty} a_n \lim_{n\to\infty} z^n = 0$. By assumption infinitely many of the a_n's are integers distinct from zero so we know that limit is not zero. This means $\lim_{n\to\infty} z^n = 0 \Rightarrow |z| < 1 \Rightarrow R = \frac{1}{1} = 1$. So the radius of convergence is at most 1.

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11. Suppose $a_n > 0, s_n = a_1 + ... + a_n$, and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges

Proof: Assume to the contrary that $\sum \frac{a_n}{1+a_n}$ converges. That means that $\sum \frac{1}{\frac{1}{a_n}+1}$ has to go to zero which means $\lim \frac{1}{a_n} = \infty \Rightarrow \lim a_n = 0$. From the Cauchy Criterion for convergence we know that there exists and $N > 0$ such that for all $n > N$, $a_n < 1$. If that is true then we can say that $a_n - \frac{a_n}{1+a_n} = \frac{a_n^2}{1+a_n} < \frac{a_n}{1+a_n}$. So that means that $\sum a_n - \frac{a_n}{1+a_n}$ converges because by assumption $\sum \frac{a_n}{1+a_n}$ converges. But that means that $\sum a_n - \frac{a_n}{1+a_n} + \sum \frac{a_n}{1+a_n} = \sum a_n$ converges. But this contradicts the assumption in the beginning of the problem that $\sum a_n$ diverges. So $\sum \frac{a_n}{1+a_n}$ diverges.

(b) Prove that

$$\frac{a_{N+1}}{s_{N+1}} + ... + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

and deduce that $\sum \frac{a_n}{s_n}$ diverges.

Proof: We know for sure that

$$\frac{s_{N+k}}{s_{N+k}} = 1$$

$$\Rightarrow \frac{s_N}{s_{N+k}} + \frac{a_{N+1} + ... + a_{N+k}}{s_{N+k}} = 1$$

$$\Rightarrow \frac{a_{N+1} + ... + a_{N+k}}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}$$

$$\Rightarrow \frac{a_{N+1}}{s_{N+k}} + ... + \frac{a_{N+k}}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}$$

Because each $a_n > 0$, if we exclude the right terms from each of the denominators on the left then we will make each term larger and thus make the left hand side larger. So

$$\frac{a_{N+1}}{s_{N+1}} + ... + \frac{a_{N+k}}{s_{N+k}} \geq 1 - \frac{s_N}{s_{N+k}}$$

To show that the $\sum \frac{a_n}{s_n}$ diverges, assume to the contrary that it converges. Then we know that for all $\epsilon > 0$ there exists an $N > 0$ such that for all $n > m > N$, $|\sum_{i=m}^{n} \frac{a_i}{s_i}| < \epsilon$. So let $\epsilon = 1 - \frac{s_{m-1}}{s_n} > 0$ then there is some $N > 0$ such $|\sum_{i=m}^{n} \frac{a_i}{s_i}| < \epsilon = 1 - \frac{s_m}{s_n} > 0$. But we have from the inequality above that $\frac{a_m}{s_m} + ... + \frac{a_n}{s_n} \geq 1 - \frac{s_{m-1}}{s_n}$ which is a contradiction. So it must be the case that $\sum \frac{a_n}{s_n}$ diverges.

(c) Prove that

$$\frac{a_n}{s_n^2} \leq \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

and deduce that $\sum \frac{a_n}{s_n^2}$ converges.

Proof: Start with the left hand side of the inequality and use the fact that $s_{n-1} < s_n$

$$\frac{a_n}{s_n^2} = \frac{s_n - s_{n-1}}{s_n s_n} \leq \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}$$

which is what we were trying to prove.

Now look at the the sums of both sides of this inequality:

$$\frac{a_1}{s_1^2} + \sum_{k=2}^{n} \frac{a_k}{s_k^2} \leq \frac{a_1}{s_1^2} + \sum_{k=2}^{n} \frac{1}{s_{k-1}} - \frac{1}{s_k}$$

The right hand side is a telescoping sum so we have that

$$\Rightarrow \frac{a_1}{s_1^2} + \sum_{k=2}^{n} \frac{a_k}{s_k^2} \leq \frac{a_1}{s_1^2} + \frac{1}{s_1} - \frac{1}{s_n}$$

$$\lim_{n\to\infty}(\frac{a_1}{s_1^2}+\sum_{k=2}^{n}\frac{a_k}{s_k^2}) \le \lim_{n\to\infty}(\frac{a_1}{s_1^2}+\frac{1}{s_1}-\frac{1}{s_n})$$

Since $a_n > 0$ that means s_n is always monotone increasing. So we can say that:

$$\sum \frac{a_n}{s_n^2} \le \frac{a_1}{s_1^2}+\frac{1}{s_1}-\lim_{n\to\infty}\frac{1}{s_n}$$

$$\Rightarrow \sum \frac{a_n}{s_n^2} \le \frac{a_1}{s_1^2}+\frac{1}{s_1} =$$

Since the partial sums of $\sum \frac{a_n}{s_n^2}$ are monotone increasing and bounded above by $\frac{a_1}{s_1^2}+\frac{1}{s_1}$ so the $\sum \frac{a_n}{s_n^2}$ converges as well.

(d) What can be said about

$$\sum \frac{a_n}{1+na_n} \quad \textit{and} \quad \sum \frac{a_n}{1+n^2a_n}?$$

Proof: For $a_n = \frac{1}{n}$ the first sum diverges while for the example in the extra credit it converges so we can not say anything about its convergence or divergence from the information given.

As for the second one it converges. To see this compare it to $\sum \frac{1}{n^2}$:

$$\sum \frac{1}{n^2} \ge \sum \frac{1}{\frac{1}{a_n}+n^2} * \frac{a_n}{a_n} = \sum \frac{1}{1+n^2a_n}$$

Since $\frac{1}{n^2}$ converges so does $\sum \frac{a_n}{1+n^2a_n}$.

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12. Suppose $a_n > 0$ and $\sum a_n$ converges. Put

$$r_n = \sum_{m=n}^{\infty} a_m.$$

(a) Prove that

$$\frac{a_m}{r_m}+...+\frac{a_n}{r_n} \ge 1-\frac{r_n}{r_m}$$

if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.

Proof: Since $\sum a_n$ converges, we know that removing a finite number of terms at the beginning does not change the convergence of the series so each r_n converges as well. So

$$\frac{r_m}{r_m} = 1$$

$$\Rightarrow \frac{a_m+...+a_{n-1}+r_n}{r_m} = 1$$

$$\Rightarrow \frac{a_m}{r_m}+...+\frac{a_{n-1}}{r_m} = 1-\frac{r_n}{r_m}$$

Because each $a_n > 0$, we know that $r_i < r_m$ for all $m < i \le n$. So changing each denominator from r_m to r_i will make the left hand side larger. So we get the desired result that

$$\frac{a_m}{r_m} + ... + \frac{a_n}{r_n} \ge 1 - \frac{r_n}{r_m}$$

To show that $\sum \frac{a_n}{r_n}$ diverges, lets assume to the contrary that it converges. Then that means for all $\epsilon > 0$ there exists an $N > 0$ such that for all $n > m > N |\sum_{i=m}^{n} \frac{a_i}{r_i}| < \epsilon$. So let $\epsilon = 1 - \frac{r_n}{r_m} > 0$ then there is some $N > 0$ such that $|\sum_{i=m}^{n} \frac{a_i}{r_i}| < \epsilon = 1 - \frac{r_n}{r_m} > 0$. But we know from the inequality above that $\frac{a_m}{r_m} + ... + \frac{a_n}{r_n} \ge 1 - \frac{r_n}{r_m}$. So it must be the case that $\sum \frac{a_n}{r_n}$ diverges.

(b) Prove that

$$\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.

Proof: Since $r_n = a_n + a_{n+1} + ...$ and $r_{n+1} = a_{n+1} + a_{n+2} + ...$ we know that $a_n = r_n - r_{n+1}$. So

$$\begin{aligned} r_n - r_{n+1} &= a_n \\ \Rightarrow a_n &= (\sqrt{r_n} - \sqrt{r_{n+1}})(\sqrt{r_n} + \sqrt{r_{n+1}}) \\ \Rightarrow \frac{a_n}{\sqrt{r_n} + \sqrt{r_{n+1}}} &= \sqrt{r_n} - \sqrt{r_{n+1}} \\ \Rightarrow \frac{a_n}{\sqrt{r_n} + \sqrt{r_n}} &= \sqrt{r_n} - \sqrt{r_{n+1}} \\ \Rightarrow \frac{a_n}{2\sqrt{r_n}} &= \sqrt{r_n} - \sqrt{r_{n+1}} \\ \Rightarrow \frac{a_n}{\sqrt{r_n}} &< 2(\sqrt{r_n} - \sqrt{r_{n+1}}) \end{aligned}$$

As in the problem before, lets look at the sums of both sides

$$\sum_{k=1}^{n} \frac{a_n}{\sqrt{r_n}} < \sum_{k=1}^{n} 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_n}{\sqrt{r_n}} < \lim_{n\to\infty} \sum_{k=1}^{n} 2(\sqrt{r_n} - \sqrt{r_{n+1}})$$

On the right hand side we again have a telescoping sum. So we continue with

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{a_n}{\sqrt{r_n}} < 2 \lim_{n\to\infty} (\sqrt{r_1} - \sqrt{r_{n+1}})$$

$$\sum \frac{a_n}{\sqrt{r_n}} < 2\sqrt{r_1}$$

So $\sum \frac{a_n}{\sqrt{r_n}}$ is monotone increasing and bounded above. So it converges by the Monotone Convergence Principle.

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13. Prove that the Cauchy product of two absolutely convergent series converges absolutely.

Proof: Let $\sum a_n$ and $\sum b_n$ are the two absolutely convergent series. Then we know that $\sum |a_n|$ and $\sum |b_n|$ both converge as well. From Merten's Theorem we know that the Cauchy Product $\sum d_n$ of $\sum |a_n|$ and $\sum |b_n|$ will converge. But this is not quite what we want. We want the Cauchy product $\sum c_n$ of $\sum a_n$ and $\sum b_n$ to converge absolutely. But we do know that $|c_n| \leq d_n$ for all n. So we have that $\sum |c_n| \leq \sum d_n \Rightarrow \sum c_n$ converges absolutely by the comparison test.

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14. If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by

$$\sigma_n = \frac{s_0 + s_1 + ... + s_n}{n+1} \qquad (n = 0, 1, 2, ...).$$

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.

Proof: If $\epsilon > 0$, we want to find an $N > 0$ such that $|\frac{(s_0 + ... + s_n)}{n+1} - l| < \epsilon$, $\forall n \geq N$. So we have that $|s_{N+1} - l| < \epsilon, |s_{N+2} - l| < \epsilon, ..., |s_{N+M+1} - l| < \epsilon$ and if we put them all together we get

$$|s_{N+1} - l| + |s_{N+2} - l| + ... + |s_{N+M+1} - l| < (M+1)\epsilon$$

$$\Rightarrow |s_{N+1} + s_{N+2} + ... + s_{N+M} - Ml| < M\epsilon$$

Also, we can say that

$$|\frac{s_{N+1} + s_{N+2} + ... + a_{N+M+1}}{N+M} - \frac{(M+1)l}{N+M+1}| < \frac{(M+1)\epsilon}{N+M+1} < \epsilon$$

As well we know that

$$|\frac{s_0 + s_1 + s_2 + ... + s_N}{N+M+1}| < \epsilon$$

and

$$|\frac{(M+1)l}{N+M+1} - l| < \epsilon$$

if we pick an appropriate N. Finally, if we put all three together we get that:

$$|\frac{s_{N+1} + s_{N+2} + ... + s_{N+M+1}}{N+M+1} - \frac{(M+1)l}{N+M+1} + \frac{s_0 + s_1 + s_2 + ... + s_N}{N+M+1} + \frac{(M+1)l}{N+M+1} - l| < 3\epsilon$$

$$\Rightarrow |\frac{s_0 + s_1 + s_2 + ... + s_{N+M+1}}{N+M+1} - l| < 3\epsilon$$

Since 3 is just a constant, we could go back and adjust our ϵ's but it won't change the proof.

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.

Proof: Look at the sequence $s_n = (-1)^n$. The sequence obviously doesn't converge because its limsup is not equal to its liminf. But $\sigma_n \to 0$ as $n \to \infty$ because we get that the $s_0 + s_1 + ... + s_n$ is equal to either 1 or -1, while the denominator goes to zero.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?

Proof: Let $a_n = \log\log(n+1)$ then we have that $\sigma_n = \frac{\log\log 1 + \log\log 2 + ... + \log\log(n+1)}{n+1} = \frac{\log(\log(1) * \log(2) * .... * \log(n+1))}{n+1} \leq \frac{\log(n \log n)}{n+1}$. The $\lim \sigma_n = 0$ but $\limsup s_n = \infty$ albeit slowly.

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4. (a) Prove that if a subsequence of a Cauchy sequence converges, then so does the original sequence.

Proof: Let $\{x_n\}$ be the Cauchy sequence, and let $\{x_{n_k}\}$ be the convergent subsequence and call its limit x. Suppose $\epsilon > 0$ then there exists an N such that $d(x_n, x_m) < \epsilon/2$ whenever $n, m > N$. Since x_{n_k} is a convergent subsequence there exists $n_0 > N$ with $d(x_{n_0}, x) < \epsilon/2$. If $m > n$ then $d(x_m - x) \leq d(x_m, x_{n_0}) + d(x_{n_0}, x) < \epsilon/2 + \epsilon/2 = \epsilon$. So $x_n \to x$.

(b) Prove that any subsequence of a convergent sequence converges.

Proof: Let $\{x_n\}$ converge to x. Then for all $\epsilon > 0$ there exists an $N > 0$ such that $d(x_n, x) < \epsilon$ for all $n > N$. Every convergent sequence is Cauchy, so that means there exists M such that $d(x_n, x_m) < \epsilon$ for all $n, m > M$. So consider a subsequence $\{x_{n_k}\}$ of the sequence. Then there exists a K such that $n_k > max\{N, M\} = J$ for all $k > K$. So then we have that $d(x_{n_k}, x) \leq d(x_{n_k}, x_J) + d(x_J, x) < \epsilon + \epsilon = 2\epsilon$. So the subsequence $\{x_{n_k}\}$ converges as well to x.

5.

(a) Prove that if $0 < a < 2$, then $a < \sqrt{2a} < 2$.

Proof: We know that $0 < a < 2 \Rightarrow 0 < a^2 < 2a < 4 \Rightarrow a < \sqrt{2a} < 2$.

(b) Prove that the sequence

$$\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, ...$$

converges.

Proof: So the sequence can be be written as $a_{n+1} = \sqrt{2a_n}$. From part (a) we know that $\sqrt{2} < \sqrt{2\sqrt{2}} < 2$, as well we know that $\sqrt{2\sqrt{2}} < \sqrt{2\sqrt{2\sqrt{2}}} < 2$. So we can guess that $a_n < \sqrt{2a_n} < 2$ $\forall n \in \mathbf{N}$. To prove this use induction. We have the case for $n = 1 : \sqrt{2} < \sqrt{2\sqrt{2}} < 2$. So now, assume its true for n and show its true for $n+1$. So $a_{n+1} = \sqrt{2a_n} < \sqrt{2\sqrt{2a_n}} < 2 \Rightarrow a_{n+1} < \sqrt{2a_{n+1}} < 2$. So we have a function that is always increasing and bounded above by 2 so the sequence converges by the Monotone Convergence Principle.

(c)Find the limit. Hint: Notice that if $\lim_{n\to\infty} a_n = l$, then $\lim_{n\to\infty} \sqrt{2a_n} = \sqrt{2l}$, by Theorem 1.

Proof: Since $\lim_{n\to\infty} a_n$ exists from part (b), then lets say that $\lim_{n\to\infty} a_n = l$. We also know that $\lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{2a_n} = \sqrt{2l}$. So we have that $l = \sqrt{2l} \Rightarrow l^2 - 2l = 0 \Rightarrow l = 0$ or $l = 2$. Since $a_n > 0$ for all n we know that $\lim_{n\to\infty} a_n = 2$.

---

6. Let $0 < a_1 < b_1$ and define

$$a_{n+1} = \sqrt{a_n b_n}, b_{n+1} = \frac{a_n + b_n}{2}.$$

(a) Prove that the sequences $\{a_n\}$ and $\{b_n\}$ each converge.

Proof: We have from the first homework that if $0 < a < b$ and $a < \sqrt{ab} < \frac{a+b}{2} < b$. We know $0 < a_1 < b_1$ so $a_1 < \sqrt{a_1 b_1} < \frac{a_1+b_1}{2} < b_1 \Rightarrow a_1 < a_2 < b_2 < b_1$. So we want to show that $a_1 < a_n < b_n < b_1$. We have that it is true for $n = 1$. Assume it is true for n, then we have that $a_1 < a_n < \sqrt{a_n b_n} < \frac{a_n+b_n}{2} < b_n < b_1 \Rightarrow a_1 < a_n < a_{n+1} < b_{n+1} < b_n < b_1 \Rightarrow a_1 < a_{n+1} < b_{n+1} < b_1$. So we have that each a_n is bounded above by any b_n and it is increasing so a_n converges by the Monotone Convergence Principle. The same is true for b_n except that it is bounded below by each a_n and decreasing so b_n converges by the Monotone Convergence Principle.

(b) Prove that they have the same limit.

Proof: From part a) we know both the sequences converge. So say the limit of $\{a_n\}$ is a, and the limit of $\{b_n\}$ is b. $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{a_n b_n}$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{a_n+b_n}{2} \Rightarrow b = \frac{a+b}{2} \Rightarrow 2b = a + b \Rightarrow a = b$. So they have the same limits.

---

8. Identify the function $f(x) = \lim_{n\to\infty}(\lim_{k\to\infty}(\cos n!\pi x)^{2k})$.

Proof: Consider the cases separately when x is irrational and when x is rational. Assume x is rational. Then $x = p/q$ for some $p, q \in \mathbf{N}$. That means $f(p/q) = \lim_{n\to\infty}(\lim_{k\to\infty}(\cos n!\pi(p/q))^{2k})$. If we let $N = 2q$, then $\forall n > N$ we will have that $\lim_{n\to\infty}(\lim_{k\to\infty}(\cos n!\pi(p/q))^{2k}) = 1$ because every value of $n!\pi(p/q)$ will be divisible by 2π if $n > N$. So that means that $\lim_{n\to\infty}(\lim_{k\to\infty}(\cos n!\pi(p/q))^{2k}) = \lim_{n\to\infty}(\lim_{k\to\infty}(1)^{2k}) = 1$.

Now consider when x is irrational. That means that for all $n \in \mathbf{N}$, $n!x$ will always be irrational. So we have that $|\cos(n!\pi x)| < 1 \Rightarrow \lim_{k\to\infty}(\cos n!\pi x)^{2k} = 0 \Rightarrow \lim_{n\to\infty}(\lim_{k\to\infty}(\cos n!\pi x)^{2k}) = 0$.

So we have that

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

---

9. Many impressive looking limits can be evaluated easily, because they are really upper or lower sums in disguise. With this remark as a hint evaluate each of the following. (Warning: one of these can be evaluated by elementary considerations.)

(iv) $\lim_{n\to\infty}(\frac{1}{n^2} + \frac{1}{(n+1)^2} + ... + \frac{1}{(2n)^2})$.

Proof: Each term in the sequence is less than $\frac{1}{n^2}$. So we can compare the limit to:

$$\lim_{n\to\infty}(\frac{1}{n^2}+\frac{1}{n^2}+...+\frac{1}{n^2}) = \lim_{n\to\infty}\frac{2n}{n^2} = 0$$

$$\Rightarrow \lim_{n\to\infty}(\frac{1}{n^2}+\frac{1}{(n+1)^2}+...+\frac{1}{(2n)^2}) \leq \lim_{n\to\infty}\frac{2n}{n^2} = 0.$$

(v) $\lim_{n\to\infty}(\frac{n}{(n+1)^2}+\frac{n}{(n+2)^2}+...+\frac{n}{(n+n)^2})$.

Proof: We can rewrite the part in the limit as an integral. To see this

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{(n+i)^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2(1+\frac{i}{n})^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{(1+\frac{i}{n})^2}\frac{1}{n}$$

If we interpret this as an integral we get:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{(1+\frac{i}{n})^2}\frac{1}{n} = \int_1^2\frac{1}{x^2}dx = \frac{1}{2}$$

(vi) $\lim_{n\to\infty}(\frac{n}{n^2+1}+\frac{n}{n^2+2^2}+...+\frac{n}{n^2+n^2})$.

Proof: Again, we can rewrite this limit as a limit of a sum:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+i^2} = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\frac{i^2}{n^2}}\frac{n}{n^2}$$

If we again interpret this as an integral we get:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{1+\frac{i^2}{n^2}}\frac{1}{n} = \int_0^1\frac{1}{1+x^2}dx = \arctan x|_0^1 = \frac{\pi}{2}$$

---

10. Although limits like $\lim_{n\to\infty}\sqrt[n]{n}$ and $\lim_{n\to\infty}a^n$ can be evaluated using facts about the behavior of the logarithm and exponential functions, this approach is vaguely dissatisfying, because integral roots and powers can be defined without using the exponential function. Some of the standard "elementary" arguments for such limits are outlined here; the basic tools are inequalities derived from the binomial theorem, notably

$$(1+h)^n \geq 1+nh, forh > 0;$$

and, for part (e),

$$(1+h)^n \geq 1+nh+\frac{n(n-1)}{2}h^2 \geq \frac{n(n-1)}{2}h^2, forh > 0.$$

(a) Prove that $\lim_{n\to\infty}a^n = \infty$ if $a > 1$, by setting $a = 1+h$, where $h > 0$.

Proof: Let $a = 1 + h$ then $\lim_{n\to\infty} a^n = \lim_{n\to\infty}(1+h)^n \geq \lim_{n\to\infty} 1 + nh = \infty$. So we have that $\lim_{n\to\infty} a^n \geq \infty \Rightarrow \lim_{n\to\infty} a^n = \infty$.

(b) Prove that $\lim_{n\to\infty} a^n = 0$ if $0 < a < 1$.

Proof: We know that $0 < a < 1 \Rightarrow 0 < 1 < \frac{1}{a}$. So then if we let $\frac{1}{a} = b$, then by part a) we have that $\lim_{n\to\infty} b^n = \infty \Rightarrow \lim_{n\to\infty} \frac{1}{b}^n = \lim_{n\to\infty} a^n = \frac{1}{\infty} = 0$.

(c) Prove that $\lim_{n\to\infty} \sqrt[n]{a} = 1$ if $a > 1$, by setting $\sqrt[n]{a} = 1 + h$ and estimating h.

Proof: So following the hint let $\sqrt[n]{a} = 1 + h \Rightarrow a = (1+h)^n \geq 1 + nh$ for all $h > 0$. So then we can solve for h to get $h \leq \frac{a-1}{n}$

$$\Rightarrow 1 < 1 + h \leq 1 + \frac{a-1}{n}$$
$$\Rightarrow \lim_{n\to\infty} 1 < \lim_{n\to\infty} (1+h) \leq \lim_{n\to\infty} (1 + \frac{a-1}{n})$$
$$\Rightarrow 1 < \lim_{n\to\infty} \sqrt[n]{a} \leq 1$$
$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{a} = 1$$

(d) Prove that $\lim_{n\to\infty} \sqrt[n]{a} = 1$ if $0 < a < 1$.

Proof: We know that $0 < a < 1 \Rightarrow 0 < 1 < \frac{1}{a}$. Then $\lim_{n\to\infty} \sqrt[n]{\frac{1}{a}} = 1 \Rightarrow \lim_{n\to\infty} \frac{\sqrt[n]{1}}{\sqrt[n]{a}} = 1 \Rightarrow \lim_{n\to\infty} \sqrt[n]{a} = 1$. So we have that

(e) Prove that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

Proof: We can use the hint to say that

$$\sqrt[n]{n} = 1 + h$$
$$\Rightarrow n = (1+h)^n \geq \frac{n(n-1)}{2} h^2$$
$$\Rightarrow \sqrt{\frac{2}{n-1}} \geq h$$

So no we have a bound for h we can say that:

$$\sqrt[n]{n} = 1 + h \leq 1 + \sqrt{\frac{2}{n-1}}$$
$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{n} \leq \lim_{n\to\infty} 1 + \sqrt{\frac{2}{n-1}}$$
$$\Rightarrow \lim_{n\to\infty} \sqrt[n]{n} \leq 1$$

Since $\sqrt[n]{n} = 1 + h$ that implies that $\lim_{n\to\infty} \sqrt[n]{n} \geq 1$ which means that $\lim_{n\to\infty} \sqrt[n]{n} = 1$.

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11. (a) Prove that a convergent sequence is always bounded.

Proof: Since the sequence converges, it is Cauchy. So that means for $\epsilon = 1$ there is an $N > 0$ such that for all $n, m > N$ we have that $d(a_n, a_m) < 1$. More importantly, this is true for all $n, m > N$ so that means $d(a_{N+1}, d_m) < 1$ for all $m > N$. So let $M = \{d(x_1, x_{N+1}), d(x_2, x_{N+1}), ..., d(x_N, x_{N+1}), 1\}$. Then $d(x_k, x_{N+1}) \leq M$ for all $k = 1, 2, 3....$

(b) Suppose that $\lim_{n\to\infty} a_n = 0$, and that each $a_n > 0$. Prove that the set of all number a_n actually has a maximum member.

Proof: Since the sequence converges to zero, we can say that there exists an N such that $|a_n - 0| < \epsilon = a_1$, for all $n > N$. Then the max of $a_1, ..., a_N$ is max of a_n for all n because all the a_n's after N will be smaller than a_1 so we need only consider the ones from 1 to N. So a_n has a maximum member.

12.

(a) Prove that

$$\frac{1}{(n+1)} < \log(n+1) - \log n < \frac{1}{n}.$$

Proof: Consider the intergral $\int_n^{n+1} \frac{1}{x}dx$. Then the smallest lower sum is just the smallest function value on the interval [0,1] evaluated at the partition $P = \{n, n+1\}$. This gives $f(n+1) * (n+1-n) = \frac{1}{n+1}$. And if we use the same partition P and just evaluate at the largest value of f we get $\frac{1}{n}$. Since the function is continuous we know the function falls somewhere in between these two values, which gives us our desired result that $\frac{1}{(n+1)} < \log(n+1) - \log n < \frac{1}{n}$.

(b) If

$$a_n = 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log n,$$

show that the sequence $\{a_n\}$ is decreasing, and that each $a_n \geq 0$. It follows that there is a number

$$\gamma = \lim_{n\to\infty}(1 + ... + \frac{1}{n} - \log n).$$

This number, known as Euler's number, has proved to be quite refractory; it is not even known whether γ is rational.

Proof: To show that the sequence is decreasing, we want to show that $a_{n+1} \leq a_n \Rightarrow 0 \leq a_n - a_{n+1}$. So we have:

$$a_n - a_{n+1} = (1 + ... + \frac{1}{n} - \log n) - (1 + ... + \frac{1}{n+1} - \log(n+1))$$

$$\Rightarrow a_n - a_{n+1} = -\log n - \frac{1}{n+1} + \log(n+1)$$

From part (a) we know that $\frac{1}{(n+1)} < \log(n+1) - \log n$, so we know have that

$$a_n - a_{n+1} = \log(n+1) - \log n - \frac{1}{n+1} > 0$$

$$\Rightarrow a_{n+1} < a_n.$$

As in part (a) we can look at the upper sum of the $\int_1^n \frac{1}{x}dx$. If we consider the partition $P = \{1, 2, 3, ..., n\}$, then the upper sum of this partition is just the function values at right endpoints because the function is monotone decreasing and continuous. Then we get that $f(2) + f(3) + ... + f(n) > \int_1^n \frac{1}{x}dx = \log n - \log 1 = \log n \Rightarrow \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log n > 0 \Rightarrow 1 + \frac{1}{2} + \frac{1}{3} + ... + \frac{1}{n} - \log n > 0 \Rightarrow a_n > 0$.

---

13. (a) Suppose that f is increasing on $[1, \infty)$. Show that

$$f(1) + ... + f(n-1) < \int_1^n f(x)dx < f(2) + ... + f(n).$$

Proof: Since $f(x)$ is increasing on the interval $[1, n]$ then this looks like upper and lower sums. The left hand side is $\sum_{k=1}^{n-1} f(x_i)\Delta x_i$ and the right hand side is $\sum_{k=2}^{n} f(x_i)\Delta x_i$ where $\Delta x_i = 1$. Since the integral is always between any upper and lower sum for a given partition if the function

$$\underline{S}(f, P) \leq \int_1^n f(x)dx \leq \overline{S}(f, P)$$

And this is exactly what we are trying to prove for any given partition of $f(x)$ where $\Delta x_i = 1$.

(b) Now choose $f = \log$ and show that

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}$$

it follows that

$$\lim_{n\to\infty} \frac{\sqrt[n]{n!}}{n} = \frac{1}{e}.$$

Proof: Since $\log x$ is increasing on $[1, \infty)$, it follows from part (a) that:

$$\log(1) + ... + \log(n-1) < \int_1^n \log(x)dx < \log(2) + ... + \log(n)$$

$$\Rightarrow \log((n-1)!) < n\log n - n + 1 < \log(n!)$$

$$\Rightarrow e^{\log((n-1)!)} < e^{n\log n - n + 1} < e^{\log(n!)}$$

$$\Rightarrow (n-1)! < n^n e^{-n+1} < n!$$

Looking at just the right inequality we have

$$n^n e^{-n+1} < n! \Rightarrow \frac{n^n}{e^{n-1}} < n!$$

Looking at just the left inequality we have

$$(n-1)! < n^n e^{-n+1} \Rightarrow n! < \frac{n^{n+1}}{e^{n-1}} < \frac{(n+1)^{n+1}}{e^n}$$

From these two separate inequalities we get what we wanted:

$$\frac{n^n}{e^{n-1}} < n! < \frac{(n+1)^{n+1}}{e^n}$$

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16. Prove that if $\lim_{n\to\infty} a_n = l$, then

$$\lim_{n\to\infty} \frac{(a_1 + ... + a_n)}{n} = l.$$

Hint: This problem is very similar to (in fact it is a special case of) Problem 13-40.

Proof: If $\epsilon > 0$, we want to find an $N > 0$ such that $|\frac{(a_1+...+a_n)}{n} - l| < \epsilon$, $\forall n \geq N$. So we have that $|a_{N+1} - l| < \epsilon, |a_{N+2} - l| < \epsilon, ..., |a_{N+M} - l| < \epsilon$ and if we put them all together we get

$$|a_{N+1} - l| + |a_{N+2} - l| + ... + |a_{N+M} - l| < M\epsilon$$

$$\Rightarrow |a_{N+1} + a_{N+2} + ... + a_{N+M} - Ml| < M\epsilon$$

Also, we can say that

$$|\frac{a_{N+1} + a_{N+2} + ... + a_{N+M}}{N+M} - \frac{Ml}{N+M}| < \frac{M\epsilon}{N+M} < \epsilon$$

As well we know that

$$|\frac{a_1 + a_2 + ... + a_N}{N+M}| < \epsilon$$

and

$$|\frac{Ml}{N+M} - l| < \epsilon$$

if we pick an appropriate N. Finally, if we put all three together we get that:

$$|\frac{a_{N+1} + a_{N+2} + ... + a_{N+M}}{N+M} - \frac{Ml}{N+M} + \frac{a_1 + a_2 + ... + a_N}{N+M} + \frac{Ml}{N+M} - l| < 3\epsilon$$

$$\Rightarrow |\frac{a_1 + a_2 + ... + a_{N+M}}{N+M} - l| < 3\epsilon$$

Since 3 is just a constant, we could go back and adjust our ϵ's but it won't change the proof.

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17. Suppose that f is continuous and $\lim_{x\to\infty} f(x+1) - f(x) = 0$. Prove that $\lim_{x\to\infty} f(x)/x = 0$. Hint: See the previous problem.

Proof: Consider the sequence of $a_n = f(n+1) - f(n)$. So we have $\lim_{x\to\infty} f(x+1) - f(x) = \lim_{x\to\infty} a_n = 0$. From the problem above we get that

$$\lim_{n\to\infty} \frac{(a_1 + ... + a_n)}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(n+1)-f(n)+f(n)-f(n-1)+...+f(2)-f(1)}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(n+1)-f(1)}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} (\frac{f(n+1)}{n} - \frac{f(1)}{n}) = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(n+1)}{n} = 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{f(n)}{n} = 0$$

So for all integers n we know that $\lim_{n\to\infty} \frac{f(n)}{n} = 0$ but that doesn't necessarily imply that $\lim_{x\to\infty} f(x)/x$ even if $f(x)$ is continuous. So I am not sure what to do from here.

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18. Suppose that $a_n > 0$ for each n and that $\lim_{n\to\infty} a_{n+1}/a_n = l$. Prove that $\lim_{n\to\infty} \sqrt[n]{a_n} = l$. Hint: This requires the same sort of argument that works in Problem 16, together with the fact that $\lim_{n\to\infty} \sqrt[n]{a} = l$, for $a > 0$.

Proof: We know that if $\lim_{n\to\infty} a_{n+1}/a_n = l$ then $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} = \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = l$. We also know that $\liminf_{n\to\infty} \frac{a_{n+1}}{a_n} \leq \liminf_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \sqrt[n]{a_n} \leq \limsup_{n\to\infty} \frac{a_{n+1}}{a_n}$, so by the Squeeze Theorem we have the desired result.

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22. (a) Use Problem 2-5 to show that if $c \neq 1$, then

$$c^m + c^{m+1} + ... + c^n = \frac{c^m - c^{n+1}}{1-c}.$$

Proof: It is easy to see that $(1-c)(1+c+c^2+...+c^{m-1}) = 1-c^m$. So then we have:

$$c^m + c^{m+1} + ... + c^n = c^m(1+c+c^2+...+c^{n-m}) = c^m(\frac{1-c^{n-m+1}}{1-c}) = \frac{c^m - c^{n+1}}{1-c}.$$

We need the fact that $c \neq 1$ because otherwise the right hand side is undefined.

(b) Suppose that $|c| < 1$. Prove that

$$\lim_{m,n\to\infty} c^m + ... + c^n = 0.$$

Proof: We know that $|c| < 1 \Rightarrow -1 < c < 1 \Rightarrow c^m < 1$ and since $m < n$, we know that $c^{n+1} < c^m \Rightarrow 0 < c^m - c^{n+1} < 1$ and we know that $0 < 1-c < 1$. So then we have that $0 < \frac{c^m - c^{n+1}}{1-c} < 1$

(c) Suppose that $\{x_n\}$ is a sequence with $|x_n - x_{n+1}| \leq c^n$, where $c < 1$. Prove that $\{x_n\}$ is a Cauchy sequence.

Proof: Lets try to use what we have from part (b) and work toward the definition of a Cauchy Sequence. Assume $n < m$, then we know that

$$|x_n - x_m| = |x_n - x_{n+1} + x_{n+1} - x_{n+2} + x_{n+2} + ... - x_{n+m-1} + x_{n+m-1} + x_m|$$

$$\Rightarrow |x_n - x_m| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + ... + |x_{n+m-2} - x_{n+m-1}| + |x_{n+m-1} + x_m|$$

$$\Rightarrow |x_n - x_m| \leq c^m + ... + c^n$$

$$\Rightarrow \lim_{m,n\to\infty} |x_n - x_m| \leq \lim_{m,n\to\infty} c^m + ... + c^n$$

$$\Rightarrow \lim_{m,n\to\infty} |x_n - x_m| \leq 0$$

$$\Rightarrow |x_n - x_m| < \epsilon$$

So we have that for all $\epsilon > 0$ $\exists N$ such that $\forall n, m > N$ $|x_n - x_m| < \epsilon$ so the sequence $\{x_n\}$ is Cauchy.

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23. Suppose that f is a function on $\mathbf{R}$ such that

$$(*)|f(x) - f(y)| \leq c|x - y|, forallxandy,$$

where $c < 1$. (Such a function is called a contraction).

(a) Prove that f is continuous.

Proof: We want to show that $\forall\epsilon > 0, \exists\delta > 0$ such that $\forall x, y \in \mathbf{R}, |x-y| < \delta \Rightarrow |f(x)-f(y)| < \epsilon$. If we pick $\delta = \frac{\epsilon}{2c}$ and from the condition above on f, we get $|f(x) - f(y)| \leq c|x - y| < c\delta = c(\frac{\epsilon}{2c}) = \frac{\epsilon}{2} < \epsilon$. So f is continuous.

(b) Prove that f has at most one fixed point.

Proof: Assume there were two distinct fixed points x, y. Then by definition $f(x) = x$ and $f(y) = y \Rightarrow |f(x) - f(y)| = |x - y|$, but we also have that $|f(x) - f(y)| \leq c|x - y|$ where $c < 1$ but this would imply that $|f(x) - f(y)| < |x - y|$ which is a contradiction. So f has at most one fixed point.

(c) By considering the sequence

$$x, f(x), f(f(x)), ...$$

for any x, prove that f does have a fixed point. (This result, in a more general setting, is known as the "contraction lemma.")

Proof: We know that $|f(f(x)) - f(x)| \leq c|f(x) - x|$ because of the given property of f. As well, $|f(f(f(x))) - f(f(x))| \leq c|f(f(x)) - f(x)| \leq c^2|f(x) - x|$. So in general we have that $|f^{n+1}(x) - f^n(x)| \leq c^n|f(x) - x|$. So if we take the limit as $n \to \infty$ we get that $|f^{n+1}(x) - f^n(x)| \leq 0 \Rightarrow f(f^n(x)) \leq f^n(x)$. So the fixed point is the limit of the sequence above.

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24. (a) Prove that if f is differentiable and $|f'| < 1$, then f has at most one fixed point.

Proof: Assume there were two distinct fixed points x, y. Then by definition $f(x) = x$ and $f(y) = y \Rightarrow |f(x) - f(y)| = |x - y| \Rightarrow \frac{|f(x)-f(y)|}{|x-y|} = 1 \Rightarrow \lim_{x \to y} |\frac{f(x)-f(y)}{x-y}| = \lim_{x \to y} 1 \Rightarrow |f'(y)| = 1$. This is a contradiction to the fact that $|f'| < 1$ for all y. So that means f has at most one fixed point.

(b) Prove that if $|f'(x)| \leq c < 1$ for all x, then f has a fixed point.

Proof: So $|f'(x)| = |\frac{f(x)-f(y)}{x-y}| \leq c < 1 \Rightarrow |f(x) - f(y)| \leq c|x - y|$. This means that this is a contraction. So if we consider the sequence from question 23 part (c), we will get the same conclusion that f has at least one fixed point, namely the limit point of the sequence.

(c) Give an example to show that the hypothesis $|f'(x)| \leq 1$ is not sufficient to insure that f has a fixed point.

Proof: Let $f(x) = x + 1$, then $f'(x) = 1$. But this is a line parallel to the line $f(x) = x$ so the two will never cross, so f has no fixed points. Actually any line of the form $f(x) = x + c$ where $c \neq 0$ will work.

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25. This problem is a sort of converse to the previous problem. Let b_n be a sequence defined by $b_1 = a, b_{n+1} = f(b_n)$. Prove that if $b = \lim_{n\to\infty} b_n$ exists and f' is continuous at b, then $|f'(b)| \leq 1$. Hint: If $|f'(b)| \leq 1 > 1$, then $|f'(b)| > 1$ for all x in an interval around b, and b_n will be in this interval for large enough n. Now consider f on the interval $[b, b_n]$.

Proof: Following the hint, assume $f'(b) > 1$. Since f' is continuous, for all $\epsilon > 0$ and $x \in (b - \epsilon, b + \epsilon)|f'(x)| > 1$. The $\lim_{n\to\infty} b_n = b$ means there exists N such that $b_n \in (b - \epsilon, b + \epsilon)$ for all $n > N$. That means f is differentiable at b, which means f is continuous at b. So from a problem above we have that$f(b) = b$. Suppose $b_n < b$ and $f' > 1$ in this interval, we have $f(b) > f(b_n)$. Then

$$|\frac{f(b) - f(b_n)}{b - b_n}| > 1 \Rightarrow f(b) - f(b_n) > b - b_n \Rightarrow b - b_{n+1} < b - b_n \Rightarrow b_{n+1} < b_n$$

This is a contradiction to the assumption that $b_n < b_{n+1} < b$. And things would follow similarly if $b_n > b$ and $f' < 1$. Also, for $b_n = b$, then $f'(b) = 0$. Thus $|f'(b)| \leq 1$.

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27. Let $\{x_n\}$ be a sequence which is bounded, and let

$$y_n = \sup\{x_n, x_{n+1}, x_{n+2}, ...\}.$$

(a) Prove that the sequence of $\{y_n\}$ converges. The limit $\lim_{n\to\infty} y_n$ is denoted $\overline{\lim}_{n\to\infty} x_n$ or $\limsup_{n\to\infty} x_n$, and called the **limit superior**, or **upper limit**, of the sequence $\{x_n\}$.

Proof: Since $\{x_n\}$ is bounded then $\{y_n\}$ is bounded as well. Since we are only removing terms as $n \to \infty$ then y_n is non-increasing. So it converges by the Monotone Convergence Principle.

(b) Find $\overline{\lim}_{n\to\infty} x_n$ for each of the following:

(i) $x_n = \frac{1}{n}$.

$\overline{\lim}_{n\to\infty} x_n = 0$

(ii) $x_n = (-1)^n \frac{1}{n}$.

$\overline{\lim}_{n\to\infty} x_n = 0$

(iii) $x_n = (-1)^n[1 + \frac{1}{n}]$.

$\overline{\lim}_{n\to\infty} x_n = 1$

(iv) $x_n = \sqrt[n]{n}$.

$\overline{\lim}_{n\to\infty} x_n = 1$

(c) Define $\underline{\lim}_{n\to\infty} x_n$ and prove that

$$\underline{\lim}_{n\to\infty} x_n \leq \overline{\lim}_{n\to\infty} x_n.$$

Proof: Let $\{x_n\}$ be as before, and let

$$y'_n = \inf\{x_n, x_{n+1}, x_{n+2}, ...\}.$$

The limit $\lim_{n\to\infty} y_n$ is denoted $\underline{\lim}_{n\to\infty} x_n$ or $\liminf_{n\to\infty} x_n$, and called the **limit inferior**, or **lower limit**, of the sequence $\{x_n\}$. Also, we know that $y'_n \leq y_n$. If we take the limit as n goes to infinity of both sides we get:

$$\Rightarrow \lim_{n\to\infty} y'_n \leq \lim_{n\to\infty} y_n$$

$$\Rightarrow \lim_{n\to\infty} \inf\{x_n, x_{n+1}, x_{n+2}, ...\} \leq \lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, ...\}$$

$$\Rightarrow \underline{\lim}_{n\to\infty} x_n \leq \overline{\lim}_{n\to\infty} x_n.$$

(d) Prove that $\lim_{n\to\infty} x_n$ exists iff $\overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n$ and that in this case $\lim_{n\to\infty} x_n = \overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n$.

Proof:

($\Rightarrow$) Assume $\lim_{n\to\infty} x_n$ exists. From above we have that $\underline{\lim}_{n\to\infty} x_n \leq \overline{\lim}_{n\to\infty} x_n$. So it is enough to prove that $\overline{\lim}_{n\to\infty} x_n \leq \underline{\lim}_{n\to\infty} x_n$.

($\Leftarrow$) Assume $\overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n$. This means the smallest subsequential limit point is equal to the largest subsequential limit point. This means there is only one. So that means $\lim_{n\to\infty} x_n$ exists and is in fact equal to $\overline{\lim}_{n\to\infty} x_n = \underline{\lim}_{n\to\infty} x_n$

(e) Recall the definition, in Problem 8-18, of $\overline{\lim} A$ for a bounded set A. Prove that if the numbers x_n are distinct, then $\underline{\lim}_{n\to\infty} x_n = \overline{\lim} A$, where $A = \{x_n : n \in \mathbf{N}\}$
