Math 230a HW3 Extra Credit

Erik Lewis

November 15, 2006

1 Extra Credit

---

16. Regard Q, the set of all rational numbers, as a metric space, with $d(p,q) = |p-q|$. Let E be the set of all $p \in Q$ such that $2 < p^2 < 3$. Show that E is closed and bounded in Q, but that E is not compact. Is E open in Q?

Proof: We can more accurately describe E as $\{x : x \in Q \cap ((-\sqrt{3}, -\sqrt{2}) \cup (\sqrt{2}, \sqrt{3}))\}$.

Pick any point $x \in E$. Then it is easy to see that $-3 < x < 3$ so E is bounded. As well, let x_0 be a limit point of E. Since $Q \subset R$, if we can find a set $A \subset R$ such that $A \cap Q = E$ then E is closed in R. So let $A = [-\sqrt{3}, -\sqrt{2}] \cup [\sqrt{2}, \sqrt{3}]$. Then $A \cap Q = E$ and since A is closed in R, E is closed in Q.

To show this is not compact we can use something similar to the proof in question 14. We must find a cover that has no finite subcover. Lets just look at the positive part of E and find a cover of that half with no finite subcover. So construct a set $G_n = \{(\sqrt{2}, a - \frac{1}{2n}) \cup (a + \frac{1}{2n}, \sqrt{3}) : \sqrt{2} < a < \sqrt{3}, a \in \mathbf{Q}^c\}$. Then our collection $\{G_n\}$ will be an open cover of everything in $E \cap R$ except the irrational number a. This will have no finite subcover because if it did there would be some largest value of n such that there was a smallest fraction $\frac{1}{n}$ such that the interval $(a - \frac{1}{2n}, a + \frac{1}{2n})$ would contain infinitely many rational numbers in E not covered by the suggested finite subcover of $\{G_n\}$. So E is not compact.

Yes E is open in Q. Pick a point $p \in E$. Then there exists a $B(p,r)$ where $r = min\{|-\sqrt{3} - p|, |-\sqrt{2} - p|, |\sqrt{3} - p|, |\sqrt{2} - p|\}$ such that $B(p, \frac{r}{2}) \subset E$.

---

17. Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in [0,1]? Is E compact? Is E perfect?

Proof: Without repeating Cantor's argument in its entirety, we assume that E is countable and try to list everything as such:

$$a_1 = 0.a_{11}a_{12}a_{13}...$$

$$a_2 = 0.a_{21}a_{22}a_{23}...$$

$$\vdots$$
$$a_n = 0.a_{n1}a_{n2}a_{n3}...$$
$$\vdots$$

Then we can always construct a number $y = 0.y_1y_2y_3...$ where $y_i = 7$ if $a_{ii} = 4$ and $y_i = 4$ if $a_{ii} = 7$. This number y is in E and is obviously not on our list which contradicts the fact that E is countable. So E must be uncountable.

E is not dense in [0,1] because there is no number in $(0.4\overline{7}, 0.7\overline{4}) \cap E$.

E is compact. To prove this we must show that E is closed and bounded. E is obviously bounded above by 1 and below by 0. Pick a limit point p of E such that $p \notin E$. Then $p = p_1p_2p_3...p_k...$ and there is some smallest value of k such that $p_k \neq 4$ and $p_k \neq 7$. So even if this is the only place where p and some point $x \in E$ differ, p will always be at least $\frac{1}{10^{k+1}}$ away from $x \forall x \in E$. We could do a case by case argument to show that if $p_k = 0, 1, 2, 3, 5, 6, 8, or 9$ and finding specific neighborhoods for each case, but it is true that the neighborhood of radius $\frac{1}{10^{k+1}}$ will always work. So this means that p is not a limit point of E because $B(p, \frac{1}{10^{k+1}}) \subset E^c$. So that means any point not in E is not a limit point of E so that means every limit point of E is in E so E is closed. Since E is closed and bounded it is compact.

E is perfect. To prove this pick a point $p \in E$ and show p is a limit point of E. Let $p = p_1p_2p_3....$ Then we can find a sequence that approaches p by picking successive x_n's so that p and x_n are the same for the first n decimal places then differ at the nth decimal place and then do whatever afterward like are all 4's or are the same everywhere but at the nth decimal place. Then $lim_{n\to\infty} d(x_n, p) = 0$ and each x_n is distinct and not equal to p.

---

18. Is there a nonempty perfect set of R^1 which contains no rational number?

Proof: Yes. Consider the set E constructed in the following manner. First take a closed interval in R^1 with irrational endpoints call it $A = [a_{irr}, b_{irr}]$. Since $\mathbf{Q}$ is countable we know that we can enumerate all of the ones between a_{irr} and b_{irr}. So assume $F = \{q_1, q_2, ..., q_n, ...\}$ is a list of all the rationals between a_{irr} and b_{irr}. Then for q_1 remove a small open interval (a_1, b_1) around q_1 such that $a_{irr} < a_1 < q_1 < b_1 < b_{irr}$. Let $A_1 = A \backslash (a_1, b_1)$.

To define A_n first determine if we already removed q_n. If so then do nothing and move on to q_{n+1} and just let $A_n = A_{n-1}$. If not, then define $A_n = A_{n-1} \backslash (a_n, b_n)$ where $a_{irr} < a_n < q_n < b_n < b_{irr}$ and $a_n, b_n \in \mathbf{Q}^c$. We have to pick them a little more carefully at the nth step though. Let $q_n - a_n < \max{}_{i=1}^{n}\{|q_n - b_i|\}$ and $q_n - b_n < \max{}_{i=1}^{n}\{|q_n - a_i|\}$. This way we do not accidentally remove any endpoints so that A_n stays closed.

So let $E = \bigcap_{n=1}^{\infty} E_n$. Since each E_n is closed and bounded, they are compact. So the intersection is compact, so E is compact. By construction it contains no rational number.

All that is left to show is that E is perfect. That means every point of E is a limit point. So pick some $p \in E$. Then for all $\epsilon > 0$ we can find a rational q_i such that $p < q_k < p + \epsilon$. From our construction, we removed q_i at some point and so it fell between some (a_i, b_i). Since $p \in E$ it must be the case that $p < a_k < q_k$. By construction, $a_k \in E$ as well. That means we found a point $a_k \neq p$ of E within ϵ of p. So p is a limit point of E and we are done.

---

19. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

Proof: Since A is closed we know that $\overline{A} \cap B = A \cap B = \emptyset$. As well, because B is closed we have that $A \cap \overline{B} = A \cap B = \emptyset$. These two statements along with the assumption that $A \cap B = \emptyset$ means that A and B are separated.

(b) Prove the same for disjoint open sets.

Proof: Assume that $A \cap B = \emptyset$ but that either $\overline{A} \cap B \neq \emptyset$ or $A \cap \overline{B} \neq \emptyset$. Without loss of generality assume that $\overline{A} \cap B \neq \emptyset$. Then we know that there is some $p \in \overline{A} \cap B$, and since $A \cap B = \emptyset$, this point p must be in A'. Since $p \in B$ and B open we know that $\exists r_0 > 0$ such that $B(p, r) \subset B$. Also, $p \in A' \Rightarrow \forall r > 0 \exists B(p, r), q$ such that $q \in B(p, r) \cap A$. So that means there is a $q \in B(p, r_0) \cap A$ but we know that $B(p, r_0) \subset B \Rightarrow q \in B \Rightarrow A \cap B \neq \emptyset$ which is a contradiction. So we know that $\overline{A} \cap B = A \cap B = \emptyset$. It follows similarly that $A \cap \overline{B} \neq \emptyset$. So we have that A and B are separated if they are disjoint open sets as well.

(c) Fix $p \in X, \delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly with $>$ in place of $<$. Prove that A and B are separated.

Proof: First, show that A and B are both open sets. So first pick $x \in A$. Then $d(x, p) < \delta$ which means $\exists B(x, s)$ where $0 < s < \delta - d(x, p)$. So now pick some $z \in B(x, s)$. Then

$$d(z, p) \leq d(z, x) + d(x, p) \leq s + d(x, p) \leq \delta - d(x, p) + d(x, p) = \delta.$$

So $z \in A$. So we have that $\forall x \in A, B(x, s) \subset A \Rightarrow A$ is open. Similarly, pick $x \in B$. Then $d(x, p) > \delta$ which means $\exists B(x, s)$ where $0 < s < d(x, p) - \delta$. So now pick some $z \in B(x, s)$. Then

$$d(z, p) \geq d(p, x) - d(x, z) \geq d(x, z) - s > d(p, x) - (d(p, x) - \delta) = \delta.$$

So $d(p, z) > r$ which means $z \in B$. So we have that $\forall x \in B, B(x, s) \subset B \Rightarrow B$ is open.

A and B are obviously disjoint because the distance metric is well defined so a point can't both be less than δ from p and greater than δ from p at the same time. So we know that $A \cap B = \emptyset$. We have two disjoint open sets, so from part (b) we know that A and B are separated.

(d)Prove that every connected metric space with at least two points is uncountable.

Proof: Suppose the two points are p and q. Since they are distinct $d(p, q) = r > 0$. Pick some $0 < r_1 < r$ and assume that there is no point x such that $d(p, x) = r_1$. Then we can separate the space into two sets A and B where A and B are the same as in part (c). So there must be a point x such that $d(p, x) = r_1$. If we let r_1 vary through all of the real numbers in the interval (0,r), we will get uncountably many points.

---

20. Are closures and interiors of connected sets always connected?

Proof: The interiors of connected sets are not always connected. Take the set $E \subset R^2$ where $E = \{B(p, 1) \cup y - axis \cup B(q, 1) : p = (0, -3), q = (0, 3) \in R^2\}$. The interior of this set is just the two open balls $B(p, 1) \cup B(q, 1)$ but this is obviously not connected by exercise 19 because we have two disjoint open balls so they are separated so they form a separation of E^0, and E^0 is thus not connected.

The closures of connected sets are always connected. Suppose we have a connected set E such that its closure is disconnected, $\overline{E} = A \cup B$ where A and B are a separation. Since E is connected we must have that $A \cap E = \emptyset$ or $B \cap E = \emptyset$ because otherwise E would be separated by A and B as well. So say that $A \cap E$ is empty. Then that means $E \subseteq B \Rightarrow \overline{E} \subseteq \overline{B}$. Since E is connected $A \cap \overline{B} = \emptyset$. So $A = A \cap (A \cup B) = A \cap \overline{E} \subseteq A \cap \overline{B} = \emptyset$. Since $A = \emptyset$, $\overline{E}$ must be connected.

---

21. Let A and B be separated subsets of some R^k, suppose $\mathbf{a} \in A, \mathbf{b} \in B$, and define

$$p(t) = (1 - t)\mathbf{a} + t\mathbf{b}$$

for $t \in R^1$. Put $A_0 = \mathbf{p}^{-1}(A), B_0 = \mathbf{p}^{-1}(B)$.

(a) Prove that A_0 and B_0 are separated subsets of R^1.

Proof: We want to show that $\overline{A}_0 \cap B_0 = \emptyset$ and that $\overline{B}_0 \cap A_0 = \emptyset$. To prove that $\overline{A}_0 \cap B_0 = \emptyset$, assume to the contrary that $\overline{A}_0 \cap B_0 \neq \emptyset$ then pick $t_0 \in A_0 \cap B_0$. If $t_0 \in B_0$ then we have that $\mathbf{p}(t_0) \in A \cap B$ which is a contradiction to the fact that A and B are separated. So it must be the case that $t_0 \in B_0'$. If we can show that $t_0 \in B_0'$ implies $\mathbf{p}(t_0) \in B'$ then we are done because we come to the same contradiction again. So for all $\epsilon > 0$ there exists a δ so that $\forall t \in B(t_0, \delta)$, $\mathbf{p}(t) \in B(\mathbf{p}(t_0), \epsilon)$. So then pick $\delta = \frac{\epsilon}{|b-a|}$. Since t_0 is a limit point there is some $t_1 \in B(t_0, \delta) \cap B_0$ such that $t_1 \neq t_0$. So then we have that $\mathbf{p}(t_1) \in \mathbf{p}(B(t_0, \delta)) \Rightarrow \mathbf{p}(t_1) \in B(\mathbf{p}(t_0), \frac{\epsilon}{|b-a|}(b - a)) = B(\mathbf{p}(t_0), \epsilon)$. So we have that $\mathbf{p}(t_1) \neq \mathbf{p}(t_0)$ for any ϵ-ball around $\mathbf{p}(t_0) \Rightarrow \mathbf{p}(t_0) \in B'$. That means that $\mathbf{p}(t_0) \in A \cap \overline{B}$ which is a contradiction to the fact that A and B are separated. So $A_0 \cap \overline{B}_0 = \emptyset$ and similarly $\overline{A}_0 \cap B_0 = \emptyset$ which means A_0 and B_0 are separated.

(b) Prove that there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$

Proof: Assume then there is no t_0, such that $\mathbf{p}(t_0) \notin A \cup B$. Then $\forall t_0, \mathbf{p}(t_0) \in A$ or $\mathbf{p}(t_0) \in B$ but it cannot be in both because A and B are separated. So that means that $(0, 1) \subset \mathbf{p}^{-1}(A \cup B) = A_0 \cup B_0$. Since $(0, 1)$ is connected it must be completely contained in A_0 or B_0. So assume that $(0, 1) \subset A_0$. That means $[0,1] \in \overline{A}_0$, but $\mathbf{p}(1) \in B$ so that is a contradiction because A_0 and B_0 are separated by the proof of part (a). So that means there exists some $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.

(c)Prove that every convex subset of R^k is connected.

Proof: Suppose we have some set E that is a convex subset of R^k that is not connected. Then there exist a separation of E, where $E = A \cup B$ and A and B are separated. Then there exists $t_0 \in (0, 1)$ such that $\mathbf{p}(t_0) \notin A \cup B$ from part (b). But this is an immediate contradiction to the fact that E is convex because by definition $\mathbf{p}(t_0) \in E$ and $E = A \cup B$. So that means E is connected.

---

27. Define a point p in a metric space X to be a condensation point of a set $E \subset X$ if every neighborhood of p contains uncountably many points of E. Suppose $E \subset R^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable.

Proof: Let $\{V_n\}$ be a countable base for R^k, let W be the union of those $\{V_n\}$ for which $E \cap V_n$ is at most countable, and show that $P = W^c$.

To show P is perfect, we must show that every point is a limit point. So assume that $p \in P$. If p is not a limit point then there exists $B(p,r)$ such that $B(p,r) \cap E$ is empty. That is a contradiction to the fact that p is a condensation point. So every point p is a limit point of P. So P is perfect.

($\Leftarrow$) Suppose $x \in W^c = \bigcap_{i=1}^{\infty} V_i^c$. We want to show $x \in P$. Assume to the contrary that $x \notin P$. Then there exists some $r_0 > 0$ such that $B(x, r_0) \cap E$ contains only finitely many points of E. Since $B(x, r_0)$ is an open set it must be equal to a union of V_α taken from our countable base $\Rightarrow (\bigcup_{i=1}^{\infty} V_i) \cap E$ is countable $\Rightarrow (V_1 \cap E) \cup (V_2 \cap E) \cup ... \subseteq W$. This is a contradiction to the fact that $x \in W^c$.

($\Rightarrow$) Suppose $x \in P$ and $x \notin W^c$. Then $x \in W \Rightarrow x \in V_j$ for some j where $V_j \cap E$ is countable. then $B(x, r_0) \subset V_j$. But we know since $x \in P$ that for all $r B(x,r) \cap E$ has uncountably many points which is a contradiction to the fact that $B(x, r_0) \subset V_j$ has at most countably many points. So $P = W^c$ and we are done.

---

28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable.

Proof: Let $E \subset X$ where X is the separable metric space and E is closed. If E is countable we are done, because $E = P \cup E$, where $P = \emptyset$. So assume that E is uncountable. Then let P be the set of all condensation points of E. From question 27, P is perfect and since E is closed $P \subset E$ and there are only countably many points in E that aren't in P. So then we have $E = P \cup (E - P)$ where P is perfect and $E - P$ is countable, and we are done.

---

29. Prove that every open set in R^1 is the union of an at most countable collection of disjoint segments. Hint in book

Proof: Suppose $G \subset R^1$ is an open set. Then for each $x \in G$ define a set $I_x = \bigcup\{I : I \text{ open interval}, x \in I, I \subset G\}$ and call this set a maximum connected component of G corresponding to x. I_x is open, but we still have to show I_x is connected. So let $a, b \in I_x$ with $a < b$ and let $a < c < b$, prove $c \in I_x$, then that will mean I_x is an interval. Let $x' \in G, x \neq x'$ then $I_x = I_{x'}$ or $I_x \cap I_{x'} = \emptyset$. To show this, suppose $I_x \cap I_{x'} \neq \emptyset$. Then look at $I_x \cup I_{x'}$. It is an open interval that contains both x and x'. So that means $I_x \cup I_{x'} \subset I_x$ because $I_x \cup I_{x'}$ participates in the union that defines I_x. As well though, $I_x \subset I_x \cup I_{x'} \Rightarrow I_x = I_x \cup I_{x'}$ and similarly $I_{x'} = I_x \cup I_{x'} \Rightarrow I_x = I_{x'}$. It is clear that $G = \bigcup_{x \in G} I_x = \bigcup_{I disjoint} I_x$. Since these intervals are disjoint, we have from the last homework that we can pick a unique rational in each I_x so that we get a function from the disjoint intervals into Q which means we have countably many disjoint intervals. Therefore, G is a countable union of disjoint open intervals.

---

30. Imitate the proof of Theorem 2.43 to obtain the following result: If $R^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset of R^k, then at least one F_n has a nonempty interior. Equivalent statement: If G_n is a dense open subset of R^k, for $n = 1, 2, 3, ...$, then $\cap_1^\infty G_n$ is not empty (in fact, it is dense in R^k).

Proof: Let G_n be a dense open subset of R^k for $n = 1, 2, 3, ...$ We will try to construct a sequence $\{V_n\}$ of neighborhoods as follows.

Suppose G_o is a nonempty open subset of R^k. Since G_1 is a dense subset of R^k, we have that $G_0 \cap G_1$ is nonempty. So we can pick some $x_1 \in G_0 \cap G_1$. Since we have the intersection of two open sets we know that $G_0 \cap G_1$ is open as well. Then we can find some open set $V_1 \subset G_0 \cap G_1$ such that $\overline{V}_1$ is also contained in $G_0 \cap G_1$. Again, since V_1 is a nonempty open set and G_2 is a dense open set their intersection is open and nonempty, so construct nonempty V_2 such that $\overline{V}_2 \subset V_1 \cap G_2$. Then $V_2 \subset V_1$. We can continue in this manner with V_{n-1} a nonempty open set and G_n a dense open set such that their intersection is nonempty. Then we construct V_n such that $\overline{V}_n \subset V_{n-1} \cap G_n$. Each $\overline{V}_n \subset \overline{V}_{n-1}$, so we have a nested sequence of closed sets $\overline{V}_1 \supset V_2 \supset ... \supset V_n \supset ....$ So we have that $V = \bigcap_{n=1}^\infty \overline{V}_n \neq \emptyset$. So for any open subset G_0 we have that $G_0 \cap V \neq \emptyset$ which means that not only is the set nonempty but it is dense in R^k.
