Math 230a Extra Credit for Homework 4

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1 Extra Credit

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1. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, ...$ by the recursion formula

$$x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}).$$

Prove that $\{x_n\}$ decreases monotonically and that $\lim_{n\to\infty} x_n = \sqrt{\alpha}$.

Proof: Start with what we know:

$$\sqrt{\alpha} < x_1$$
$$\Rightarrow \alpha < x_1^2$$
$$\Rightarrow x_1^2 + \alpha < x_1^2 + x_1^2$$
$$\Rightarrow \frac{x_1^2 + \alpha}{x_1} < 2x_1$$
$$\Rightarrow \frac{1}{2}(x_1 + \frac{\alpha}{x_1}) < x_1$$
$$\Rightarrow x_2 < x_1$$

This same process follows for all n and $n+1$. So we have that the sequence is monotone decreasing. Now, we show by induction that $\sqrt{\alpha} < x_n$ for all $n \in \mathcal{N}$. We know that $\sqrt{\alpha} < x_1$. Now assume $\sqrt{\alpha} < x_n$ and we want to show it is true for x_{n+1}, so assume to the contrary that $x_{n+1} \leq \sqrt{\alpha}$ then:

$$x_{n+1} \leq \sqrt{\alpha}$$
$$\Rightarrow \frac{1}{2}(x_n + \frac{\alpha}{x_n} \leq \sqrt{\alpha}$$
$$\Rightarrow (x_n + \frac{\alpha}{x_n}) \leq 2\sqrt{\alpha}$$
$$\Rightarrow x_n^2 + \alpha \leq 2\sqrt{\alpha}x_n$$

$$\Rightarrow x_n^2 - 2\sqrt{\alpha}x_n + \sqrt{\alpha} \leq 0$$
$$\Rightarrow (x_n^2 - \sqrt{\alpha})^2 \leq 0$$
$$\Rightarrow (x_n - \sqrt{\alpha})^2 = 0$$
$$\Rightarrow x_n = \sqrt{\alpha}$$

This contradicts our induction hypothesis that $\sqrt{\alpha} < x_n$. So it must be the case that $x_{n+1} > \sqrt{\alpha}$.

So we have that x_n is monotone decreasing and bounded below by $\sqrt{\alpha}$. This means that x_n converges. Let x be the limit of the sequence $\{x_n\}$. Then $\lim x_n = \lim x_{n+1} = x$, so we have:

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{1}{2}(x_n + \frac{\alpha}{x_n})$$
$$\Rightarrow x = \frac{1}{2}(x + \frac{\alpha}{x})$$
$$\Rightarrow 2x - x = \frac{\alpha}{x}$$
$$\Rightarrow x^2 = \alpha$$
$$\Rightarrow x = \sqrt{\alpha}, -\sqrt{\alpha}$$

Since x_n is bounded below by $\sqrt{\alpha}$ for all n, we know $\lim x_n$ can not be $-\sqrt{\alpha} \Rightarrow \lim_{n\to\infty} x_n = \sqrt{\alpha}$.

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2. If $a_1 = \alpha, a_2 = \beta, a_{n+2} = \frac{1}{2}(a_{n+1} + a_n), n = 1, 2, ...,$ prove that

(a) $\{a_n\}$ converges.

Proof:

(b) $\lim_{n\to\infty} a_n = \frac{\alpha+2\beta}{3}$.

Proof: We know that

$$2a_3 = a_2 + a_1$$
$$2a_4 = a_3 + a_2$$
$$\vdots$$
$$2a_{n+2} = a_{n+1} + a_n$$

Now if we add up all these equalities we get

$$2a_3 + 2a_4 + ... + 2a_{n+2} = a_1 + 2a_2 + ... + 2a_n + a_{n+1}$$
$$\Rightarrow 2a_{n+2} + 2a_{n+1} = a_1 + 2a_2 + a_{n+1}$$
$$\Rightarrow 2a_{n+2} + a_{n+1} = a_1 + 2a_2$$

Now take the limit of both sides

$$\lim_{n\to\infty} 2a_{n+2} + a_{n+1} = \lim_{n\to\infty} a_1 + 2a_2$$

$$3 \lim_{n\to\infty} a_n = \alpha + 2\beta$$

$$\lim_{n\to\infty} a_n = \frac{\alpha + 2\beta}{3}$$

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3. If $\{s_n\}$ is a real sequence, define its arithmetic mean σ_n by

$$\sigma_n = \frac{s_0 + s_1 + ... + s_n}{n+1}$$

for $n \geq 1$. Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that

$$s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k.$$

Assume that $\lim_{n\to\infty} n a_n = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges.

Proof: First show that $s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k$. We can begin with the fact that:

$$s_n - \sigma_n = \frac{(n+1)s_n}{n+1} - \frac{s_0 + s_1 + ... + s_n}{n+1} = \frac{n s_n - s_0 - s_1 - ... - s_{n-1}}{n+1}$$

$$\Rightarrow s_n - \sigma_n = \frac{s_1 - s_1 - s_0 - s_1 - ... - s_{n-1} + n s_n}{n+1}$$

$$\Rightarrow s_n - \sigma_n = \frac{s_1 - s_0 - 2s_1 - s_2 - ... - s_{n-1} + n s_n}{n+1}$$

$$\Rightarrow s_n - \sigma_n = \frac{a_1 - 2s_1 - s_2 - ... - s_{n-1} + n s_n}{n+1}$$

If we continue this process, adding and subtracting $i s_i$ we get:

$$\Rightarrow s_n - \sigma_n = \frac{a_1 + 2a_2 + ... - n s_{n-1} + n s_n}{n+1}$$

$$\Rightarrow s_n - \sigma_n = \frac{a_1 + 2a_2 + ... + n s_n - n s_{n-1}}{n+1}$$

$$\Rightarrow s_n - \sigma_n = \frac{a_1 + 2a_2 + ... + n a_n}{n+1}$$

$$\Rightarrow s_n - \sigma_n = \frac{1}{n+1} \sum_{k=1}^{n} k a_k$$

Now we use this result to show that $\{s_n\}$ converges. Since $n a_n \to 0$ we know that there exists an N such that $n a_n < \epsilon$. So then we have that

$$\frac{1}{n+1} \sum_{k=m}^{n} k a_k < (\frac{n-m+1}{n+1})\epsilon < \epsilon$$

So $\sum na_n$ satisfies the Cauchy Criterion and thus converges. So $s_n = \frac{1}{n+1}\sum ka_k + \sigma_n$. Since both pieces on the right hand side converge, then their sum converges to the sum of their limits, which means $\{s_n\}$ converges.

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4. Let $0 < x_1 < y_1$ and let

$$x_{n+1} = \frac{x_n + y_n}{2} \qquad y_{n+1} = \sqrt{x_n y_n} \qquad n \in \mathbf{N}.$$

(a) Prove $0 < y_n < x_n$ for all $n \in \mathbf{N}$.

Proof: We have from the first homework that if $0 < a < b$ and $a < \sqrt{ab} < \frac{a+b}{2} < b$. From this, we know $0 < y_1 < x_1$ so

$$y_1 < \sqrt{x_1 y_1} < \frac{x_1 + y_1}{2} < x_1$$
$$\Rightarrow y_1 < y_2 < x_2 < x_1.$$

So we want to show that $y_1 < y_n < x_n < x_1$. We have that it is true for $n = 1$. Assume it is true for n, then we have that

$$y_1 < y_n < \sqrt{x_n y_n} < \frac{x_n + y_n}{2} < x_n < x_1$$
$$\Rightarrow y_1 < y_n < y_{n+1} < x_{n+1} < x_n < x_1$$
$$\Rightarrow y_1 < y_{n+1} < x_{n+1} < x_1.$$
$$\Rightarrow 0 < y_{n+1} < x_{n+1}.$$

(b) Prove y_n is increasing and bounded above, and x_n is decreasing and bounded below.

Proof: From above we have that each y_n is bounded above by any x_n and it is increasing so y_n converges by the Monotone Convergence Principle. The same is true for x_n except that it is bounded below by each y_n and decreasing so x_n converges by the Monotone Convergence Principle.

(c) Prove that $0 < x_{n+1} - y_{n+1} < \frac{x_1 - y_1}{2^n}$ for $n \in \mathbf{N}$.

Proof: From part (a) we have that $y_{n+1} < x_{n+1} \Rightarrow 0 < x_{n+1} - y_{n+1}$ for all $n \in \mathbf{N}$. Now we show the other half of the inequality with induction. For $n = 1$ we have that $x_2 - y_2 < \frac{x_1 - y_1}{2}$

$$x_2 < x_1$$
$$x_2 - y_2 < x_1 - \sqrt{x_1 y_1}$$

Since we showed above that $\sqrt{x_1 y_1} < \frac{x_1 + y_1}{2}$, we get

$$x_2 - y_2 < x_1 - \frac{x_1 + y_1}{2}$$

$$x_2 - y_2 < \frac{x_1 + y_1}{2}$$

Now we assume this is true for n and show its true for $n + 1$:

$$
\begin{aligned}
y_n &< x_n \\
\Rightarrow y_n^2 &< x_n y_n \\
\Rightarrow y_n &< \sqrt{x_n y_n} \\
\Rightarrow 2y_n &< 2\sqrt{x_n y_n} \\
\Rightarrow y_n + y_n - 2\sqrt{x_n y_n} &< 0 \\
\Rightarrow y_n + x_n - 2\sqrt{x_n y_n} &< x_n - y_n \\
\Rightarrow \frac{y_n + x_n}{2} - \sqrt{x_n y_n} &< \frac{x_n - y_n}{2} \\
\Rightarrow x_{n+1} - y_{n+1} < \frac{x_n - y_n}{2} &< \frac{1}{2}\frac{x_1 - y_1}{2^{n-1}} = \frac{x_1 - y_1}{2^n}
\end{aligned}
$$

(d) Prove that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

Proof: If we take the limit as n goes to infinity of part (c), we get:

$$\lim_{n\to\infty} x_{n+1} - y_{n+1} < \lim_{n\to\infty} \frac{x_1 - y_1}{2^n}$$

$$\lim_{n\to\infty} x_{n+1} - y_{n+1} \leq 0$$

Since $x_n > y_n$ for all n we know that

$$\lim_{n\to\infty} x_{n+1} - y_{n+1} = 0$$

$$\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} y_{n+1}$$

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5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence with $\lim_{n\to\infty} a_n = A \neq 0$. Suppose $a_n \neq 0$ for all $n \in \mathbf{N}$. Show that the two series

$$\sum_{n=1}^{\infty} |a_{n-1} - a_n| \qquad \textit{and} \qquad \sum_{n=1}^{\infty} |\frac{1}{a_{n-1}} - \frac{1}{a_n}|$$

either both converge or both diverge.

Proof: First, rewrite the second series with a common denominator:

$$|\frac{1}{a_{n-1}} - \frac{1}{a_n}| = |\frac{a_n - a_{n-1}}{a_{n-1}a_n}|$$

Now use the limit comparison test:

$$\lim_{n\to\infty} \frac{|a_{n-1} - a_n|}{|\frac{a_n - a_{n-1}}{a_{n-1}a_n}|} = \lim_{n\to\infty} |a_{n-1}a_n| = A^2 \neq 0$$

Since the limit exists and is less that ∞, we know by the limit comparison test that either both series converge or both diverge.

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6. Assume that $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$. Let $r_n = \sum_{m=n}^{\infty} a_m$. Prove that

$$\frac{a_m}{r_m} + ... + \frac{a_n}{r_n} > 1 - \frac{a_n}{r_n}$$

if $m < n$ and then prove that $\sum_{n=1}^{\infty} \frac{a_n}{r_n}$ diverges.

Proof: Since $\sum a_n$ converges, we know that removing a finite number of terms at the beginning does not change the convergence of the series so each r_n converges as well. So

$$\frac{r_m}{r_m} = 1$$

$$\Rightarrow \frac{a_m + ... + a_{n-1} + r_n}{r_m} = 1$$

$$\Rightarrow \frac{a_m}{r_m} + ... + \frac{a_{n-1}}{r_m} = 1 - \frac{r_n}{r_m}$$

Because each $a_n > 0$, we know that $r_i < r_m$ for all $m < i \leq n$. So changing each denominator from r_m to r_i will make the left hand side larger. So we get the desired result that

$$\frac{a_m}{r_m} + ... + \frac{a_n}{r_n} \geq 1 - \frac{r_n}{r_m}$$

To show that $\sum \frac{a_n}{r_n}$ diverges, lets assume to the contrary that it converges. Then that means for all $\epsilon > 0$ there exists an $N > 0$ such that for all $n > m > N |\sum_{i=m}^{n} \frac{a_i}{r_i}| < \epsilon$. So let $\epsilon = 1 - \frac{r_n}{r_m} > 0$ then there is some $N > 0$ such that $|\sum_{i=m}^{n} \frac{a_i}{r_i}| < \epsilon = 1 - \frac{r_n}{r_m} > 0$. But we know from the inequality above that $\frac{a_m}{r_m} + ... + \frac{a_n}{r_n} \geq 1 - \frac{r_n}{r_m}$. So it must be the case that $\sum \frac{a_n}{r_n}$ diverges.

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7. Let $f : X \to X$ be a function from a metric (X, d) into itself. A point $p \in X$ is called a fixed point of f if $f(p) = p$. The function f is called a contraction of X if there is a positive number $\alpha < 1$ such that

$$d(f(x), f(y)) \leq \alpha d(x, y)$$

for all $x, y \in X$. Prove that

(a) a contraction of any metric space X is uniformly continuous on X;

Proof: We want to show that for all $\epsilon > 0$ there exists a $\delta > 0$ such that $d(f(p), f(q)) < \epsilon$ for all $p, q \in X$ when $d(p, q) < \delta$. So if we pick $\delta = \frac{\epsilon}{2\alpha}$ then we get:

$$d(f(p), f(q)) \leq \alpha d(p, q) \leq \alpha(\frac{\epsilon}{2\alpha}) = \frac{\epsilon}{2} < \epsilon$$

So a contraction is uniformly continuous on any X.

(b) a contraction f of a complete metric space X has a unique fixed point p (Fixed-point theorem).

Proof: Consider the sequence $x, f(x), f(f(x)), ...$ Then we have that $d(f(f(x)) - f(x)) \leq \alpha d(f(x) - x)$ because of the given property of X. As well, $d(f(f(f(x))) - f(f(x))) \leq \alpha d(f(f(x)) -$

$f(x)) \leq \alpha^2 d(f(x) - x)$. So in general we have that $d(f^{n+1}(x) - f^n(x)) \leq \alpha^n d(f(x) - x)$. So if we take the limit as $n \to \infty$ we get that $d(f^{n+1}(x) - f^n(x)) \leq 0 \Rightarrow d(f(f^n(x)) - f^n(x)) = 0$ because distances are non-negative. But from this we have that $f(f^n(x)) = f^n(x)$ as $n \to \infty$. So the fixed point is the limit of the sequence above.

To show the fixed point is unique, assume not. Then there are two fixed points $p \neq q$. Then we have that $f(p) = p$ and $f(q) = q$. Now look at the distance between $f(p)$ and $f(q)$:

$$d(f(p), f(q)) \leq \alpha d(p, q)$$

$$d(p, q) \leq \alpha d(p, q)$$

$$1 \leq \alpha$$

This is a contradiction to the assumption that $0 < \alpha < 1$. So it must be the case that $p = q$, which means that the fixed point is unique.

(c) Does the fixed point theorem still hold if the above contraction condition is replaced with $d(f(x), f(y)) \leq d(x, y)$?

Proof: Consider $X = \mathcal{R}$. Then the function $f(x) = x + 1$ has no fixed point even though $d(f(x), f(y)) = d(x, y)$ for all $x, y \in \mathcal{R}$. So we need the strict inequality to guarantee a fixed point.

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8. (a) True or False: (Create partial results as many as possible)

i. $\limsup(x_n + y_n) = \limsup x_n + \limsup y_n$.

ii.$\limsup(x_n y_n) = \limsup x_n \limsup y_n$.

Proof:

i. We know from the first homework that

$$\sup\{x_n + y_n, x_{n+1} + y_{n+1}, ...\} \leq \sup\{x_n, x_{n+1}, ...\} + \sup\{y_n, y_{n+1}, ...\}$$

$$\Rightarrow \lim_{n\to\infty} \sup\{x_n + y_n, x_{n+1} + y_{n+1}, ...\} \leq \lim_{n\to\infty} (\sup\{x_n, x_{n+1}, ...\} + \sup\{y_n, y_{n+1}, ...\})$$

$$\Rightarrow \lim_{n\to\infty} \sup\{x_n + y_n, x_{n+1} + y_{n+1}, ...\} \leq \lim_{n\to\infty} \sup\{x_n, x_{n+1}, ...\} + \lim_{n\to\infty} \sup\{y_n, y_{n+1}, ...\}$$

$$\Rightarrow \limsup(x_n + y_n) \leq \limsup x_n + \limsup y_n$$

ii. It follows similarly that

$$\sup\{x_n y_n, x_{n+1} y_{n+1}, ...\} \leq \sup\{x_n, x_{n+1}, ...\} \sup\{y_n, y_{n+1}, ...\}$$

$$\Rightarrow \lim_{n\to\infty} \sup\{x_n y_n, x_{n+1} y_{n+1}, ...\} \leq \lim_{n\to\infty} (\sup\{x_n, x_{n+1}, ...\} \sup\{y_n, y_{n+1}, ...\})$$

$$\Rightarrow \lim_{n\to\infty} \sup\{x_n y_n, x_{n+1} y_{n+1}, ...\} \leq \lim_{n\to\infty} \sup\{x_n, x_{n+1}, ...\} \lim_{n\to\infty} \sup\{y_n, y_{n+1}, ...\})$$

$$\Rightarrow \limsup(x_n y_n) \leq \limsup x_n \limsup y_n$$

(b) State and prove the similar results for $\liminf$.

Proof:

(c) What can you say about those results in (a)(i) if one of the $\{x_n\}$ or $\{y_n\}$ converges, say $\lim x_n = x$?

Proof: If one of the two sequences converge then we can get that $\limsup(x_n+y_n) \geq \limsup x_n + \limsup y_n \Rightarrow \limsup(x_n + y_n) = \limsup x_n + \limsup y_n$.

(d) What can you say about those results in (a)(ii) if one of the $\{x_n\}$ or $\{y_n\}$ converges, say $\lim x_n = x > 0$ and if $x_n, y_n \geq 0$ for all n?

Proof: With these conditions, we can say that $\limsup(x_n y_n) = \limsup x_n \limsup y_n$

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9. Let $\{x_n\}$ be a sequence of real numbers and let $a_n = \frac{x_1+x_2+\ldots x_n}{n}$.

(a) Prove that

$$\liminf x_n \leq \liminf a_n \leq \limsup a_n \leq \limsup x_n.$$

(b) Find an example for which all of the limits in part (a) are finite and all of the inequalities are strict.

Proof:

(c) Find an example for which some of the limits are infinite and others are finite.

Proof: Look at the sequence

$$a_n = \begin{cases} \frac{-n}{2} & \text{if } n \text{ is even} \\ \frac{n+1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Then we get $\liminf x_n = -\infty, \liminf a_n = 0, \limsup a_n = 1/2, \limsup x_n = \infty \Rightarrow -\infty < 0 < \frac{1}{2} < \infty$.

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10. Consider the sequence $\{b_n\}$ we constructed in the proof of the theorem that the monotone convergence principe implies the completeness axiom. Prove

(a) $\{b_n\}$ is monotone increasing and bounded above. Hence, by the monotone convergence principle we have $\lim_{n\to\infty} b_n = b$ for some $b \in \mathcal{R}$.

(b) Prove that $b_n \leq b$ for all n.

(c) Prove that $b = \sup S$.

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11. Prove the following seven theorems are all equivalent in $\mathcal{R}$: ...

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12. Find an alternating series $\sum(-1)^n a_n$ that does not converge, even though $a_n \to 0$.

Proof: Define a piecewise function

$$a_n = \begin{cases} 1/n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

If we pick a_n this way, then $\lim a_n = 0$, but the $\sum(-1)^n a_n = \sum_{n=1}^{\infty} \frac{1}{2n}$ which diverges.

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13. Prove that $\sum \frac{\sin nx}{n}$ converges for any $x \in \mathcal{R}$.

Proof: Look $\cos(nx) + i\sin(nx) = e^{inx}$. So then we know that $\sum \frac{\cos(nx)}{n} + \frac{i\sin(nx)}{n} = \sum \frac{e^{inx}}{n} = \sum \frac{(e^{ix})^n}{n}$. If we let $z = e^{ix}$ then we can use the ratio test to find the radius of convergence:

$$\sum \frac{(e^{ix})^n}{n} = \sum \frac{z^n}{n}$$

$$\Rightarrow \lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} |\frac{\frac{z^{n+1}}{n+1}}{\frac{z^n}{n}}| = \lim_{n\to\infty} |\frac{nz}{n+1}| = |z|$$

So when $|z| < 1$ the $\sum \frac{z^n}{n}$ converges which means the real part and the imaginary part of $\sum \frac{\cos(nx)}{n} + \frac{i\sin(nx)}{n}$ each converge independently. If we consider just the imaginary part, then $\sum \frac{\sin(nx)}{n}$ converges if $|\sin(nx)| < 1$ which is true for all $x \in \mathcal{R}$ except when $x = \frac{\pi}{2} + \pi n$. So we must check these values of x on the boundary.

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14. Prove that $\sum \frac{\cos nx}{n}$ converges if x is not of the form $2k\pi$ for any integer k.

Proof: We can use some of the same work from question 13 above, to get that $\sum \frac{\cos nx}{n}$ will converge as long as $|z| < 1$. We must again look at the boundary. The only time this will not converge if $x = 2k\pi$ because we get $\sum \cos(nx) = \sum \cos(n2k\pi) = \sum 1 = \infty$. So we only have a problem when

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15. True or false: the Cauchy product of two convergent series is convergent.

Proof: False. Consider the series $\sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$ which converges. Then we have that the Cauchy product is:

$$c_1 = 1$$
$$c_2 = -(\frac{2}{\sqrt{2}})$$
$$c_3 = \frac{2}{\sqrt{3}} + \frac{1}{2}$$
$$\vdots$$

So if we continue in this manner then a more general term for c_n is:

$$c_n = \sum_{k=0}^{n} \frac{(-1)^n}{\sqrt{n-k+1}\sqrt{k+1}}$$

$$\Rightarrow (n-k+1)(k+1) = (\frac{n}{2}+1)^2 - (\frac{n}{2}-k)^2 \leq (\frac{n}{2}+1)^2$$

$$\Rightarrow |c_n| \geq \sum_{k=0}^{n} \frac{2}{n+2} = \frac{2(n+1)}{n+2}$$

$$\Rightarrow \lim_{n\to\infty} |c_n| = \infty$$

Since $\lim_{n\to\infty} |c_n| = \infty$ we get that $\lim_{n\to\infty} c_n$ doesn't go to zero which means $\sum c_n$ diverges.