Math 230A Homework

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1 Zhou Questions

1. If $\{1, 2, ..., n\} \sim \{1, 2, ..., m\}$, prove that $m = n$.

Proof: Assuming $m, n \in \boldsymbol{N}$ then by trichotomy exactly one of the following is true: $n < m, n = m$ or $n > m$. So assume $n < m$, then we must find an equivalence function f from $\{1, 2, ..., n\}$ to $\{1, 2, ..., m\}$ that is 1-1 and onto. For f to be 1-1 we must map 1 to any of the elements in $\{1, 2, ..., m\}$ and then 2 to any of the remaining $m - 1$ elements because f has to be 1-1 so it can not take 2 to whatever 1 was mapped to. And then 3 must be taken to the remaining $m - 2$ elements because 3 can not be taken to the first two elements. We continue in this manner assigning elements from the first set and mapping them to elements in the second set. Since $n < m$, once we get to the last step where we have to map n to one of the remaining $m-(n-1)$ elements we will still have $m-n$ elements with no pre-image left but this means the function is not onto which is a contradiction, so $n \geq m$.

Assume instead that $n > m$, then again we must find an equivalence function. Since again the function must be 1-1 we will have the first m elements in the set $\{1, 2, ..., n\}$ going in the same arbitrary way as outlined above onto the set $\{1, 2, ..., m\}$. Then we will still have $n - m$ elements in $\{1, 2, ..., n\}$ that have to be sent to repeat elements in $\{1, 2, ..., m\}$. So f will be onto $\{1, 2, ..., m\}$ as we wanted but it will not be one to one because it will necessarily have to take two elements in $\{1, 2, ..., n\}$ to one element in $\{1, 2, ..., m\}$. So no such function exists and $n \leq m$.

So, by trichotomy the only possibility left is that $n = m$.

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2. Let S be a finite set containing n elements and let T be the collection of all subsets of S. Show that T is a finite set and find the number of elements in T.

Proof: Name the elements of S as a_i where a_i is the ith element of the set S. Then a possible way to think about constructing an arbitrary subset $t \in T$ is by either including element a_i or not including it in t. So think of each subset as a string of 0's and 1's where a 1 means we include that element and a 0 means we do not include that element. Then all the possible subsets t can be thought of as all the possible binary strings of length n and there are 2^n. So $|T| = 2^n$ which obviously implies that the number of elements of T is finite.

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3. Find a 1-1 function f from N onto S where S is the set of all odd numbers explicitly.

Proof: Let $f : N \to S$ be defined as:

$$f(n) = \begin{cases} n & \text{if } n \text{ is odd} \\ -(n-1) & \text{if } n \text{ is even} \end{cases}$$

With this function $n \in \{1, 3, 5, 7 \dots\}$ will be taken to themselves which are all the positive numbers in S. And if $n \in \{2, 4, 6, 8 \dots\}$ then we shift one back to an odd number and then take the opposite to ensure we hit all the negative odd numbers. So our function is definitely onto. As well this function is obviously 1-1, but to be more rigorous lets consider all the possibilities. If we had $f(n_1) = f(n_2)$ then either n_1 and n_2 were both even, both odd, or one was even and one was odd. If n_1, n_2 both even then that means $n_1 = n_2$ which is trivial. If n_1, n_2 both odd then we get $-(n_1 - 1) = -(n_2 - 1) \implies n_1 - 1 = n_2 - 1 \implies n_1 = n_2$. Last, one was even and one was odd. Without loss of generality let n_1 be even and n_2 be odd. Then we have $n_1 = -(n_2 - 1)$ but this means that n_1 is a negative number which is obviously a contradiction if $n_1 \in N$. This means it isn't even possible that one was even and one was odd. And the only other possibilities lead to f being 1-1. So we have an equivalence function from $N \to S$ which means S is countable.

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4. Let P_n be the set of all polynomials of degree n with integer coefficients. Prove that P_n is countable.

Proof: Well each $p \in P_n$ is uniquely determined by its coefficients. So we have that each p can be thought of as an n-tuple of $\mathbf{Z} \times \mathbf{Z} \times \ldots \times \mathbf{Z} \backslash \{0\}$ where there are n $\mathbf{Z}$'s and 1 $\mathbf{Z} \backslash \{0\}$ in the cross. From Rudin Theorem 2.13, if A is a countable set, and let B_n be the set of all n-tuples $(a_1, ..., a_n)$, where $a_k \in A$ $(k = 1, ..., n)$ and the elements need not be distinct, then B_n is countable. Here we have $A = P_n$ and so $B_n = \mathbf{Z} \times \mathbf{Z} \times \ldots \times \mathbf{Z} \backslash \{0\}$ is countable because it is the finite cross of countable sets.

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5. Use the above result to prove that the set of all polynomials with integer coefficients is a countable set.

Proof: Let P_k be the k-th P_n where each P_k is countable from problem 4. If we put all these P_k's together we will have all the polynomials with integer coefficients. So we have $P = \bigcup_{k=1}^{\infty} P_k$. From Rudin Theorem 2.12, we have that the union of a sequence of countable sets from n = 1, 2, 3, . . . is a countable set itself. Because every polynomial is in some P_k, we know that same polynomial will be in P itself. So we have that P is countable where P is the set of all polynomials with integer coefficients

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6. For each $p \in \mathrm{P}_n$ (P_n as defined in #4), define $B(p) = \{x : p(x) = 0\}$. Prove that union $B(p)$ is countable.

Proof: Since $p \in P_n$ we know that it has degree n and so p has at most n distinct roots by the Fundamental Theorem of Algebra. So each $B(p)$ is finite and thus are countable. So $\bigcup_{p \in P_n} B(p)$ is countable because it is a countable union of countable sets. We know the union itself is countable because from problem 4 we know that P_n is countable.

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7. Rudin: p. 43: 2, 3, 4

(2) A complex number z is said to be algebraic if there are integers $a_0, \dots, a_n$, not all zero, such that

$$a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n = 0.$$

Prove that the set of all algebraic numbers is countable.
Proof: Call the set of all algebraic numbers $\mathcal{A}$. We know from problem 6 above that $\bigcup_{p \in P_n} B(p)$ is countable. But we can see that $a_0 z^n + a_1 z^{n-1} + \dots + a_{n-1} z + a_n$ is an element of P_n. As well we can think of $B(p) = \{x : p(x) = 0\}$ as $\mathcal{A}(p) = \{p : p(z) = 0\}$. For the same reasons as above we know that each $\mathcal{A}(p)$ is finite for any given $p \in \mathcal{A}$. From question 6 then we know that $\mathcal{A}$ is countable.

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(3) Prove that there exist real numbers which are not algebraic.
Proof: Assume that all real numbers are algebraic. Then we have that the set $\mathcal{A}$ of all the algebraic numbers is equal to $\mathbf{R}$. But that means we have a countable set being the exact same set as an uncountable set. That means we could find a 1-1 and onto function $f(x) = x$ such that $\mathcal{A} \sim \mathbf{R}$. This is a contradiction because $\mathcal{A}$ is countable from the previous question while $\mathbf{R}$ is uncountable. So there must exist not just some real numbers that are not algebraic but uncountable many, because if there were any less than that, we would have that $\mathbf{R}$ was the union of two countable sets which would mean $\mathbf{R}$ was itself countable.

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(4) Is the set of all irrational real numbers countable?

No. It follows from the intuition that we can not remove a set of countable points like $\mathbf{Q}$ from an uncountable set like $\mathbf{R}$ and all of a sudden have a countable set like $\mathbf{Q}^c$.

Assume $\mathbf{Q}^c$ is countable. We also know that $\mathbf{Q}$ is countable. And we know that $\mathbf{Q} \bigcup \mathbf{Q}^c = \mathbf{R}$. But $\mathbf{R}$ is uncountable. This is a contradiction because we know that the countable union of countable sets is countable and we have the union of two countable sets, which is countable, equaling an uncountable set. So we know that the irrational numbers are uncountable because they are infinite and not countable.

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8. Show that the following sets are countable:
(a) the set of circles in the complex plane having rational radii and centers with rational coordinates.
Proof: We can completely define a circle by the two rational coordinates of its' center and its' radius. So we have an ordered triple of three rational numbers. So we can think of these circles as elements of $\mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$. From Rudin Theorem 2.13, if A is a countable set, and let B_n be the set of all n-tuples $(a_1, ..., a_n)$, where $a_k \in A$ $(k = 1, ..., n)$ and the elements need not be distinct, then B_n is countable. So in this case, B_n is the set of ordered triples of A $= \mathbf{Q}$, where A is countable. So $\mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$ is countable for sure. We must show that the function f from all the circle in the complex plane having rational radii and centers with rational coordinates to $\mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$ is 1-1. But it is obvious that any ordered triple uniquely determines a circle of rational radii in the complex plane. Since we have a 1-1 function from the circles into $\mathbf{Q} \times \mathbf{Q} \times \mathbf{Q}$ we know the set of circles in the complex plane having rational radii and centers with rational coordinates is countable.

(b) any collection of disjoint intervals of positive length.
Proof: In any interval of positive length, we know a rational number lies inside because **Q** is dense in **R**. Pick a specific collection of disjoint intervals of positive length and call it $\mathcal{I}$. Then we can construct a function f that takes every $q \in \mathbf{Q}$ to the interval $[a, b] \in \mathcal{I}$ where the rational number q falls. We know this is a function because the intervals are disjoint. If it somehow took a single $q \in \mathbf{Q}$ to two different intervals then this would imply that $q \in [a, b]$ and $q \in [c, d]$ but that would mean that $[a, b] \bigcap [c, d] = q$ which contradicts the fact that the intervals are disjoint. So we know that this is a function and since every interval of positive length contains a rational number, we know that the function is onto. So we have an onto function from a countable set **Q** to $\mathcal{I}$ so $\mathcal{I}$ must be at most countable. Since $\mathcal{I}$ is obviously not finite, it must be infinite and thus countable.

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9. Let f be a real valued function defined for every x in the interval $0 \leq x \leq 1$. Suppose there is a positive number M having the following property: for every choice of a finite number of points $x_1, \ldots, x_n$ in the interval $0 \leq x \leq 1$, the sum $|f(x_1) + \ldots + f(x_n)| \leq M$. Let S be the set of x in $0 \leq x \leq 1$ for which $f(x) \neq 0$. Prove that S is countable.
Proof: Consider the sets

$$A_1 = \{x : |f(x_i)| \geq 1\}$$

$$A_2 = \{x : 1 > |f(x_i)| \geq \frac{1}{2}\}$$

$$A_3 = \{x : \frac{1}{2} > |f(x_i)| \geq \frac{1}{3}\}$$

$$\vdots$$

$$A_n = \{x : \frac{1}{n-1} > |f(x_i)| \geq \frac{1}{n}\}$$

$$\vdots$$

If we can show each A_n is at most countable then $\bigcup_{n=1}^{\infty} A_n$ will be countable and we will just have to show that $S = \bigcup_{n=1}^{\infty} A_n$ and then we will be done. We can actually show that each A_n is finite. Assume there were infinitely many $x_i \in A_1$ such that $f(x_i) > 1$. So pick some set $\{a_1, a_2, ...a_{M+1}\}$ such that each $f(a_i) > 1$, and by the statement of the problem this subset of [0,1] must have the property that $|f(a_1) + f(a_2) + ... + f(a_{M+1})| \leq M$ but since everything in $\{a_1, a_2, ...a_{M+1}\}$ has a positive function value under f and is greater than 1 then we know that $f(a_1) + f(a_2) + ... + f(a_{M+1}) > (1 + 1 + ...1)$ and so $|f(a_1) + f(a_2) + ... + f(a_{M+1})| > (1 + 1 + ...1)$. This is a contradiction. Similarly if we assume that there were infinitely many $x_i \in A_1$ such that $f(x_i) < -1$ then we will come to the same contradiction. So the set A_1 must be finite contain at most M elements with function values greater than 1 by the Archimedean Principle and M elements with function values less than 1 by the Archimedean Principle. This proof follows similarly for each A_i except that there will be at most $2M$ elements with function values between $\frac{1}{2}$ and 1 and $2M$ elements with function values between $-\frac{1}{2}$ and -1. And this will continue on for each A_i having at most $2iM$ elements. Since each A_i is finite we know that $\bigcup_{n=1}^{\infty} A_n$ is countable. Since every element $f(x_i) \neq 0$ then we know that $|f(x_i)| > 0$ and thus x_i appears in exactly 1 of the A_i. So we know that $S = \bigcup_{n=1}^{\infty} A_n$ and S is thus countable.

10. The purpose of this problem is to show that the open interval (0,1) is equivalent to the closed interval [0,1]. In the process we will discover that both intervals are equivalent to [0,1) and (0,1]. It is then easy to generalize to any interval [a,b] with a < b. In each case, you need to find an explicit equivalence function.

Define $f : (0{,}1) \to \boldsymbol{R}$ as follows. For $n \in \boldsymbol{N}$, $n \geq 2$, $f(\frac{1}{n}) = \frac{1}{n-1}$ and for all other $x \in (0{,}1)$, $f(x) = x$.

(a) Prove that f is a 1-1 function from (0,1) into (0,1].
Proof: Assume $f(x) = f(y)$ $x, y \in (0, 1)$. If $f(x) = f(y)$ is of the form $\frac{1}{n-1}$ where $n \in \boldsymbol{N}$ then x, y came from $\boldsymbol{N}$. In this case we have:

$$\begin{aligned} f(x) &= f(y) \\ \implies \frac{1}{x-1} &= \frac{1}{y-1} \\ \implies y - 1 &= x - 1 \\ \implies x &= y \end{aligned}$$

If it was the case that $f(x) = f(y)$ $x, y \in (0, 1)$ and x, y weren't in the form $\frac{1}{n-1}$ where $n \in \boldsymbol{N}$, then it is trivial because $x = f(x) = f(y) = y \implies x = y$ So f(x) is 1-1.

(b) Prove that f is a function from (0,1) onto (0,1].
Proof: We want to show that for each $y \in (0, 1)$ there exists an $x \in (0, 1]$ such that $f(x) = y$. If y is of the form $\frac{1}{n_y - 1}$ where $n_y \in \boldsymbol{N}$ then there exists an $x = n_y$ such that $f(x) = y$. If y is not of that form then there is obviously an $x \in (0, 1)$, namely $x = y$, such that $f(x) = y$.

(c) Find a 1-1 function from [0,1) onto [0,1].
Consider the function:

$$f(x) = \begin{cases} \frac{1}{x-1} & \text{if } x = \frac{1}{n},\ n \in N \\ 0 & \text{if x = 0} \\ x & \text{otherwise} \end{cases}$$

From parts (a) and (b) we know that $f(x)$ will be 1-1 and onto from (0,1) to (0,1]. Since the domain and range in this question each have the point zero added to them, we can just take $0 \to 0$ and we will still have a 1-1 and onto function.

(d) Prove that [0,1) is equivalent to (0,1].
Proof: Consider the function:

$$f(x) = \begin{cases} 1 & \text{if x = 0} \\ x & \text{otherwise} \end{cases}$$

The function is the identity map from (0,1) to (0,1), and this restriction of the function will obviously be 1-1 and onto. As for the only point left in the domain, 0, we map that to the only point left in the range, 1. So we have that $f(x)$ is in fact 1-1 and onto.

(e) Prove that (0,1) is equivalent to [0,1].
Using parts (a) - (d) we can string together a set of equivalences to show that:

$$(0, 1) \sim (0, 1] \sim [0, 1) \sim [0, 1]$$

$$\implies (0, 1) \sim [0, 1]$$

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11. Let a,b,c and d be any real numbers such that a < b and c < d. Prove that [a,b] is equivalent to [c,d]. Find an equivalence function explicitly.

Proof: Consider a middle step of finding an equivalence function from [a,b] to [c,d], by first finding one from [a,b] to [0,1] and then from [0,1] to [c,d]. Consider first:

$$g(x) = \frac{x-a}{b-a}$$

This function from [a,b] to [0,1] takes the endpoints to endpoints, $g(a) = 0$ and $g(b) = 1$, and since the function is monotone increasing it will be 1-1 and onto for all other values as well. Also, consider:

$$h(x) = (d-c)x + c$$

This function from [0,1] to [c,d] takes endpoints to endpoints, $h(0) = c$ and $h(1) = d$. Again since the function is monotone increasing it will be 1-1 and onto for all other values as well. Now consider the composite of the function $h(g(x)) = f(x)$:

$$f(x) = (d-c)\left(\frac{x-a}{b-a}\right) + c = \left(\frac{d-c}{b-a}\right)(x-a) + c$$

Since the composition of bijective function is bijective we have an equivalence function from [a,b] to [c,d].

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12. If A is a countable set and B an uncountable set, prove that $B \backslash A \sim B$.

Proof: If the two sets are disjoint then the question is trivial. So assume that $A \subset B$ which i assume was the point of the question in the first place. If it is not, it will still follow fairly similarly if $A \bigcap B \neq \varnothing$ but is not all of A because the intersection could be at most A and would thus be countable so assume that this is the case. Since B is uncountable, that means it is infinite. So we know that B has a countable subset, call it C. Construct C so that we never pick any $c_i \in C$ such that $c_i = a_i$ for any $a_i \in A$. So we have that $A \bigcap C = \varnothing$. Also, list the elements of A and C as such:

$$A = \{a_1, a_2, a_3, ...\}$$

$$C = \{c_1, c_2, c_3, ...\}$$

Then we can construct a function from $B \backslash A$ to B such that we interleave elements of A and C:

$$f(x) = \begin{cases} c_{\frac{i}{2}} & \text{if } x = c_i \text{ and } i \text{ is even} \\ a_{\frac{i+1}{2}} & \text{if } x = c_i \text{ and } i \text{ is odd} \\ x & \text{otherwise} \end{cases}$$

For all values x in the domain not in C we have the identity function which is obviously 1-1 and onto. Otherwise, we have a countable set being mapped to two disjoint countable sets like taking the natural numbers to the integers. We know f is 1-1 because if $f(c_i) = f(c_j)$ then when $c_i, c_j \in C$ they both have indicies in C and so for the two things to be equal it must be the case that $i = j$ and so $c_i = c_j$. The notation here isn't great so without getting caught up in a bunch of notation, it is clear that this function will be 1-1 and onto. Since $C \subset B \backslash A$ by construction, we have the required equivalence function, and we reach the desired conclusion that $B \backslash A \sim B$.

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13. Prove Dedekind's theorem: A set S is infinite iff there is a proper subset A of S such that A is equivalent to S.

Proof: ($\Rightarrow$) Assume S is infinite. As in the previous problem, we know that we can then construct a countable subset $C = \{c_1, c_2, c_3, ...\}$ such that $C \subset S$. We can then construct a function f from S to $A = S \backslash \{c_1\}$ such that

$$f(x) = \begin{cases} c_{i+1} & \text{if } x = c_i \\ x & \text{otherwise} \end{cases}$$

Again, this is obviously 1-1 and onto if x is not in C because it is the identity map. Otherwise we have each c_i getting bumped down one which will be 1-1 and onto as well. This shows that $A = S \backslash \{c_1\}$, a proper subset of C, is equivalent to S.

($\Leftarrow$) Assume we have a set $A \subset S$ where $A \sim S$ and S is finite and cardinality of S is n. Since A is a proper subset, we know that the cardinality of A is strictly less than n. But we also know from question one that the cardinality of A is equal to the cardinality of S. This means that the cardinality of A is n but this is a contradiction to the fact that the cardinality of A is strictly less than n. So S must be infinite and we are done.

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14. Let $\boldsymbol{R}$ denote the set of real numbers and let S denote the set of all real-valued functions whose domain is $\boldsymbol{R}$. Show that S and $\boldsymbol{R}$ are not equivalent.

Proof: Using the hint, if $a \in \mathbf{R}$, let $g_a = f(a)$ be the real-valued function in S which corresponds to the real number a. Now define h by the equation $h(x) = 1 + g_x(x)$ if $x \in \mathbf{R}$. We want to show that $h \notin S$.

Since $f(\mathbf{R})$ is an equivalence function there must be some $b \in \mathbf{R}$ such that $f(b) = h(x)$ because f is onto. From our naming convention above that means that $f(b) = g_b(x)$. But $h(x) = 1 + g_x(x)$ for all $x \in \mathbf{R}$ which means that $h(x) \neq f(b)$ at $x = b$ because $f(b) = g_b(b) \neq 1 + g_b(b) = h(x)$. Since b was chosen arbitrarily, there exists no b such that $f(b) = h(x)$. This implies that f is not onto which is a contradiction. That means f is not an equivalence function. Since f is any arbitrary function so there exists no f that is an equivalence function from $\mathbf{R}$ to S so S is not equivalent to $\mathbf{R}$.

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15. For a set A, let $\mathcal{P}(A)$ be the set of all subsets of A. Prove that A is not equivalent to $\mathcal{P}(A)$. How is this problem related to #14?

Proof: Suppose there existed an equivalence function $f : A \to \mathcal{P}(A)$ such that $img(f) = \{B : B \subseteq A$, and there is $x \in A$ such that $f(x) = B\}$. It really doesn't matter which element we send x to so let's send it to the singleton subset $\{x\} \in \mathcal{P}(A)$.
Consider the specific function $f(x)$ that takes x to {x}. This function is clearly 1-1 from A into $\mathcal{P}(A)$. But this function will not be onto. Still it remains to show that there exists no such function $f : A \to \mathcal{P}(A)$.
Assume that such an equivalence function f exists. Now, and define $E = \{x : x \in A \text{ and } x \notin f(x)\}$.

Since f must be onto for it to be an equivalence function then there must be some $e \in A$ such that $f(e) = E$ where $E \subseteq A$ as described earlier. Then either $e \in f(e)$ or $e \notin f(e)$.
Case 1: $e \in f(e)$. We are assuming f(e) = E, so $e \in \{x : x \in A$ and $x \notin f(x)\}$ by the definition of E, which means $e \in A$ and $e \notin f(e)$. This is a contradiction to the assumption that $e \in f(e)$.
Case 2: $e \notin f(e)$. Again we are assuming f(e) = E, so $e \notin \{x : x \in A$ and $x \notin f(x)\}$ which means $e \in A$ but is not the case that $e \notin f(e)$ which is a roundabout way of saying $e \in f(e)$. This contradicts our earlier asumption that $e \notin f(e)$.
In other words, there exists no element $e \in A$ such that $f(e) = E$ for any function $f : A \to \mathcal{P}(A)$. This means there exists no onto function $f : A \to \mathcal{P}(A)$. So it is actually the case that $\mathcal{P}(A)$ is s strictly larger set than A.

Consider the $S' \subset S$ where S is as in question 14, where S' is the set of all real valued functions that takes everything in the domain to the set {0,1}. Then $S' \sim \mathcal{P}(\boldsymbol{R})$ because we can think about the inclusion or exclusion of an element in the subset of R as a 1 if it is in the subset and a 0 if not. This is similar as well to problem 2. So if we take a function $s' \in S'$ then let the equivalence function f take s' to the subset of R that it uniquely determines. Then this function is obviously onto $\mathcal{P}(\boldsymbol{R})$ but it is clearly not 1-1. So we know that S is at least as big as $\mathcal{P}(\boldsymbol{R})$. So we can use the conclusion above that a set is strictly smaller than its' own power set to say that there exists no equivalence function from $\boldsymbol{R} \to S$.