Solution for Assignment 2

1 a) If S and T are subsets of $\mathbb{N}$, then

$$|\mathbb{1}_S - \mathbb{1}_T| = \mathbb{1}_{S\Delta T}$$

hence if $S \neq T$, then $S\Delta T \neq 0 \Rightarrow$

$$\|\mathbb{1}_S - \mathbb{1}_T\|_\infty = \sup |\mathbb{1}_S - \mathbb{1}_T|(n) = \sup_{S\neq\emptyset} \mathbb{1}_{S\Delta T}(n) = 1$$

It follows that the open balls $\{B_{\frac{1}{2}}(\mathbb{1}_S) : S \subseteq \mathbb{N}\}$ are disjoint.

Suppose that D is a countable dense set in ℓ^∞.

Given $S \subseteq \mathbb{N}$, let $f_n \in D$, $f_n \to \mathbb{1}_S$. Then $\exists n_0 : n \geq n_0 \Rightarrow$ $f_{n_0} \in B_{\frac{1}{2}}(\mathbb{1}_S)$ and we see $D \cap B_{\frac{1}{2}}(\mathbb{1}_S) \neq \emptyset$.

Fix $g_S \in D \cap B_{\frac{1}{2}}(\mathbb{1}_S)$ for each $S \subseteq \mathbb{N}$, $S \neq \emptyset$. If $S \neq T$ then $g_S \neq g_T$ i.e., the map

$$\mathcal{P}(\mathbb{N}) \hookrightarrow D : S \mapsto g_S$$

is one-to-one $\Rightarrow$ card $D \geq$ card $\mathcal{P}(\mathbb{N}) > \aleph_0$, a contrad.

b) Given $a = (a_n) \in \ell^\infty$, define

$$\theta(a)(x) = \begin{cases} a_1 & 0 \leq x < \frac{1}{2} \\ a_2 & \frac{1}{2} \leq x < \frac{3}{4} \\ a_3 & \frac{3}{4} \leq x < \frac{7}{8} \\ \vdots \end{cases}$$

2. $f_0 = 1$, $f_1 = x$, $f_2 = x^2$ $\qquad \|1\|_2^2 = \int_{-1}^1 1^2\,dx = 2$

$$e_0 = \frac{f_0}{\|f_0\|_2} = \frac{1}{\sqrt{2}}$$

$$f_1 \cdot e_0 = x \cdot \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}\int_{-1}^1 x\,dx = 0 \qquad \|f_1\|^2 = \int_{-1}^1 x^2\,dx = \frac{2}{3}$$

$$e_1 = \frac{f_1 - (f_1\cdot e_0)e_0}{\|f_1 - (f_1\cdot e_0)e_0\|_2} = \frac{f_1}{\|f_1\|} = \sqrt{\frac{3}{2}}\,x$$

$$e_2 = \frac{f_2 - (f_2\cdot e_0)e_0 - (f_2\cdot e_1)e_1}{\|f_2 - (f_2\cdot e_0)e_0 - (f_2\cdot e_1)e_1\|} = \sqrt{\frac{5}{2}}\cdot\frac{1}{2}(3x^2 - 1)$$

$$x^2 = c_0 e_0 + c_1 e_1 + c_2 e_2$$

$$c_0 = x^2 \cdot e_0 = \int_{-1}^{1} \frac{x^2}{\sqrt{2}} dx = \frac{1}{\sqrt{2}} \left[\frac{x^3}{3} \right]_{-1}^{1} = \frac{2}{3} \cdot \frac{1}{\sqrt{2}}$$

$$c_1 = x^2 \cdot e_1 = \int_{-1}^{1} x^2 \sqrt{\frac{3}{2}}\, x\, dx = 0 \qquad (x^3 \text{ is odd})$$

$$c_2 = x^2 \cdot e_2 = \int_{-1}^{1} x^2 \sqrt{\frac{5}{2}} \cdot \frac{1}{2} (3x^2 - 1)\, dx$$

$$= \frac{1}{2} \sqrt{\frac{5}{2}} \int_{-1}^{1} (3x^4 - x^2)\, dx$$

$$= 2 \cdot \frac{1}{2} \sqrt{\frac{5}{2}} \int_{0}^{1} (3x^4 - x^2)\, dx$$

$$= \sqrt{\frac{5}{2}} \left[\frac{3x^5}{5} - \frac{x^3}{3} \right]_0^1 = \sqrt{\frac{5}{2}} \left(\frac{3}{5} - \frac{1}{3} \right)$$

$$= \sqrt{\frac{5}{2}}\ \frac{9-5}{15} = \sqrt{\frac{5}{2}} \cdot \frac{4}{15}$$

$$x^2 = \frac{2}{3\sqrt{2}} e_0 + \sqrt{\frac{5}{2}} \cdot \frac{4}{15} e_2$$

$$\|x^2\|_2^2 = \frac{2}{9} + \left(\sqrt{\frac{5}{2}} \cdot \frac{4}{15} \right)^2 = \frac{2}{9} + \frac{5}{\cancel{2}} \cdot \frac{\overset{8}{\cancel{16}}}{225} = \frac{50+40}{225}$$

$$= \frac{90}{225} = \frac{10}{25} = \frac{2}{5}$$

Check $\quad \int_{-1}^{1} |x^2|^2\, dx = \left[\frac{x^5}{5} \right]_{-1}^{1} = \frac{2}{5}$

3. $\quad \sum \frac{1}{n} e_n$ converges in H because $\sum \left(\frac{1}{n}\right)^2 < \infty$
but $\sum \|\frac{1}{n} e_n\| = \sum \frac{1}{n} = \infty$

4. See Monday's lecture notes.

5. c) We have

$$f(x) = x \sim \sum_{n \neq 0} \frac{(-1)^n}{n} i\, e^{inx}$$

Let $g(x) = \int_{-\pi}^{x} f(t)\,dt$. We have $g'(x) = f(x)$

$$g(\pi) = \int_{-\pi}^{\pi} x\,dx = 0 = g(-\pi)$$

hence $g \in C^1_{per}([-\pi,\pi])$. We have

$$g \sim \sum c_n e^{inx} \Rightarrow f = g' \sim \sum (in) c_n e^{inx}$$

(see the discussion of $C^1_{per}([-\pi,\pi])$), and thus

$$(in) c_n = i\frac{(-1)^n}{n} \Rightarrow c_n = \frac{(-1)^n}{n^2} \qquad (n \neq 0)$$

We have

$$g(x) = \int_{-\pi}^{x} t\,dt = \frac{t^2}{2}\Big]_{-\pi}^{x} = \frac{x^2-\pi^2}{2}$$

$$c_0 = \int_{-\pi}^{\pi} g(x)dx = \frac{1}{2}\int_{-\pi}^{\pi} x^2 dx - \frac{\pi^2}{2}\int_{-\pi}^{\pi} 1$$

$$= \frac{\pi^3}{3} - \pi^3 = \left(-\frac{2}{3}\right)\pi^3$$

$$\underbrace{\sum_{n\neq 0}}\ |c_n|^2 + |c_0|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{x^2-\pi^2}{2}\right]^2 dx$$

hence $2\sum \frac{1}{n^4} = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{x^2-\pi^2}{2}\right]^2 dx - |c_0|^2$ etc.