Assignment 2

1. Show that $\ell^\infty = \ell^\infty(\mathbb{N})$ and $L^\infty([0,1],\lambda)$ are not separable Banach spaces. Hint: for the first space consider $\|\mathbf{1}_S - \mathbf{1}_T\|_\infty$ for subsets $S, T \subseteq \mathbb{N}$. For the second show that there is an isometric linear mapping of the first into the second.

2. a) Use the Gram-Schmidt orthogonalization procedure on the vectors $1, x, x^2$ in $L^2([-1,1],\mathcal{B},\lambda)$ (the resulting functions e_0, e_1, e_2 are the first three "normalized Legendre polynomials").

b) Use othonormality to find the coefficients a_k in the expansion $x^2 = a_0e_0 + a_0e_1 + a_2e_2$.

c) Calculate $\|x\|_2$ directly and then by using the expansion in (b).

3. Suppose that e_n $(n \in \mathbb{N})$ is an orthonormal basis in a Hilbert space H. Show that there exist an $x \in H$ such that the corresponding expansion $x = \sum c_n e_n$ is not absolutely convergent.

4. a) Suppose that $f \in L^2_{\mathbb{R}}([-\pi,\pi],\lambda/2\pi)$ and $f(x) \sim \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx + \sum b_n \sin nx$ is its (L^2-convergent) Fourier series. Derive the usual formulae for a_n and b_n. Note: you can use the fact that we know $e_n(x) = e^{inx}$ is an orthonormal basis for $L^2_{\mathbb{C}}([-\pi,\pi],\lambda/2\pi)$.

b) What can you say about the coefficients a_n if $f(-x) = f(x)$ (i.e., f is even) or if $f(-x) = -f(x)$ (i.e., f is odd)?

5. a) Find the complex Fourier series (i.e., $\sum c_n e^{inx}$) for $f(x) = x$.

b) Use a) to evaluate $\sum \frac{1}{n^2}$.

c) Calculate $\sum \frac{1}{n^4}$ by considering the indefinite integral $\int f(x)dx$.