Math 33A Midterm 2 Study Guide

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1 Overview

The second midterm exam will cover all the material in the lectures (including the material from the first midterm because the course is cumulative). However, it will emphasize material which corresponds to sections 3.4 and 5.1-5.4 from your textbook. Below is a list of new definitions, techinques, and formulae that you should know and understand:

1. Know the definition of coordinates and coordinate vectors for a basis $(v_1, \ldots, v_m)$ of a subspace V of $\mathbb{R}^n$. This is the content of Definition 3.4.1 in Bretscher.
2. Know that coordinates are linear (i.e. Fact 3.4.2)
3. Know how to find the matrix that, when multiplied by a vector in standard coordinates for $\mathbb{R}^n$, gives you the coordinate vector of a vector in a given different basis of $\mathbb{R}^n$.
4. Know how to find the matrix of a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ with respect to a new basis $v_1, \ldots, v_n$. This is definition 3.4.3.
5. Know the relationship between the standard matrix of a linear transformation and the matrix of a linear transformation with respect to a basis $\mathcal{B}$. This is Fact 3.4.4.
6. Know how to change coordinates between a basis $\mathcal{B} = (v_1, \ldots, v_n)$ and another basis $\mathcal{C} = (w_1, \ldots, w_n)$ for vectors $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ in $\mathbb{R}^n$ which are given in terms of the standard basis, $e_1, \ldots, e_n$.
7. Know what it means for two matrices to be similar. This is definition 3.4.5.
8. Know when two vectors are considered to be orthogonal.
9. Know the definition of a unit vector.
10. Know the definition of an orthonormal set of vectors.
11. Know that orthonormal vectors are linearly independent.
12. Know that any collection of n orthonormal vectors is a basis for $\mathbb{R}^n$
13. Know the formula for orthogonal projection onto a subspace, if you are given an orthonormal basis for that subspace. This is fact 5.1.5.
14. Know that if you have an orthonormal basis $u_1, \ldots, u_n$ of $\mathbb{R}^n$ that $x = (u_1 \cdot x)u_1 + \ldots + (u_n \cdot x)u_n$ for all x in $\mathbb{R}^n$.
15. Know the definition of the orthogonal complement of a subspace along with fact 5.1.8.
16. Know the Pythagorean theorem for vectors in $\mathbb{R}^n$ along with fact 5.1.10.
17. Know the Cauchy-Schwarz inequality.

18. Know how to perform the Gram-Schmidt orthonormalization process on a set of linearly independent vectors.

19. Know how perform the QR-factorization algorithm on a matrix whose columns are linearly independent.

20. Know the definition of an orthogonal transformatio and an orthogonal matrix.

21. Know that orthogonal transformations preserve the orthogonality of a finite collection of orthogonal vectors (by fact 5.3.2).

22. Know why orthogonal matrices preserve the angles between vectors.

23. Know fact 5.3.3

24. Know that the product of two orthogonal matrices is orthogonal and that the inverse of an orthogonal matrix is orthogonal.

25. Know the definition of the transpose of a matrix.

26. Know what a symmetric matrix is and what a skew-symmetric matrix is.

27. Know that for two column vectors v and w in $\mathbb{R}^n$ that $v \cdot w = v^T w$.

28. Know that the transpose of an orthogonal matrix is its own inverse

29. Know Fact 5.3.8

30. Know Fact 5.3.9

31. Know that if $u_1, \ldots, u_m$ is an orthonormal basis for a subspace V of $\mathbb{R}^n$ that for the matrix Q, whose columns are the vectors $u_1, \ldots, u_m$, QQ^t represents orthogonal projection onto V.

32. Know that $(\text{im}A)^\perp = \ker(A^T)$

33. Know Fact 5.4.2 and 5.4.3.

34. Know the definition of the least squares solution to a linear system $Ax = b$.

35. Know what the normal equation is for a least-squares solution. This is Fact 5.4.5.

36. Know that if $A^T A$ is invertible that there is a **unique** least-squares solution given by the formula$x^* = (A^T A)^{-1} A^T b$. Your book doesn't seem to emphasize this, but if $A^T A$ is **not** invertible, then there are **infinitely** many least-squares solutions. In this case you have to solve the normal equation by row reduction to find all the least-squares solutions.

37. Know that you can circumvent Gram-Schmidt and QR-factorization to find the orthogonal projection matrix onto a subspace with the following formula: if $v_1, \ldots, v_m$ are linearly independent vectors in $\mathbb{R}^n$ that span a subspace V and A is the matrix whose columns are $v_1, \ldots, v_m$ in order, then $\text{proj}_V = A(A^T A)^{-1} A^T$. This, however, still requires that you find the inverse of a $m \times m$ matrix, which could be hard if m is large.

38. Know how to find the line that best approximates, in the sense of least squares, a collection of points in the xy-plane.

39. Know how to find the equation of a parabola that best approximates, in the sense of least squares, a collection of points in the xy-plane.

40. Know how to find the equation of an degree n polynomial that best approximates a collection of m points, where $m > n$.

2 Practice problems

The actual exam will be like the first midterm exam: it will have 5 problems, the first of which is a true/false problem. Following the true/false question I have tried to make the problems increasingly difficult. You may not agree with this assessment, but I'm mentioning this so you can budget your time and energy when taking the exam.

1. Let

$$\vec{v_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{v_3} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

.

Since $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent, they form a basis $\mathcal{B}$ of $\mathbb{R}^3$. If $\mathcal{S} = (\vec{e_1}, \vec{e_2}, \vec{e_3})$ is the standard basis, then you should be able to calculate the $\mathcal{B}$ coordinates $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ of a vector $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ in standard coordinates. Since $\vec{x} = S[\vec{x}]_{\mathcal{B}}$ we know that $[\vec{x}]_{\mathcal{B}} = S^{-1}\vec{x}$ where S is the matrix whose columns are v_1, v_2, v_3.

Let

$$\vec{w_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{w_2} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$$

$$\vec{w_3} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

. Now $\mathcal{C} = (\vec{w_1}, \vec{w_2}, \vec{w_3})$ is another basis for $\mathbb{R}^3$. We let T denote the matrix whose columns are $\vec{w_1}, \vec{w_2}, \vec{w_3}$.

You should be able to compute the matrix which represents a change of coordinates from $\mathcal{C}$-coordinates to $\mathcal{B}$-coordinates. This is just given by $S^{-1}T$ since T takes you from $\mathcal{C}$-coordinates to $\mathcal{S}$-coordinates, and then S^{-1} takes you from $\mathcal{S}$-coordinates to $\mathcal{B}$-coordinates. This is an important type of problem, so be sure you understand all the steps.

2. The questions in 5.1 are a bit ad-hoc, so I'll assume that you know all the definitions and concepts, but I don't particularly like the problems from that section.

3. From 5.2 you should be able to do 7,13,14,21,27,28. Essentially, you'd better know how to do Gram-Schmidt and QR-factorization in an actual example with numbers.

4. From 5.3 you should be able to do 5,6,7,8,9,10,11 (which are basically true/false questions). Similarly, you should be able to answer problems 13-26.

5. From 5.4 you should start by reading example 3, example 4, and example 5 carefully. Then do problems 30-32. After that really look at fact 5.4.5. Review, problem 10 in 5.4 also.