Math 110A Homework #1

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1. Problem 1.1.1: Find the quotient and remainder when a is divided by b:

   (a) $a = 302, b = 19$

   (b) $a = -302, b = 19$

   (c) $a = 0, b = 19$

   Answer:

   (a) $302 = 15 \cdot 19 + 17$, so $q = 15$ and $r = 17$

   (b) $-302 = -16 \cdot 19 + 2$, so $q = -16$ and $r = 2$. Note that $q \neq -16$ and $r \neq -17$ because the division algorithm states that the remainder, r must be less than b *and* nonnegative.

   (c) $0 = 0 \cdot 19 + 0$, so $q = 0$ and $r = 0$.

2. Problem 1.2.1 Find the greates common divisors

   (a) $(56, 72) =?$

   (b) $(24, 138) =?$

   (c) $(143, 227) =?$

   Answer:

   (a)

$$\begin{aligned} 72 &= 1 \cdot 56 + 16 \\ 56 &= 3 \cdot 16 + 8 \\ 16 &= 2 \cdot 8 + 0 \end{aligned}$$

   Thus $(56, 72) = 8$ by the Euclidean algorithm.

   (b)

$$\begin{aligned} 138 &= 5 \cdot 24 + 18 \\ 24 &= 1 \cdot 18 + 6 \\ 18 &= 3 \cdot 6 + 0 \end{aligned}$$

   Thus $(24, 138) = 6$ by the Euclidean algorithm.

(c)

$$\begin{aligned}
227 &= 1 \cdot 143 + 84 \\
143 &= 1 \cdot 84 + 59 \\
84 &= 1 \cdot 59 + 23 \\
59 &= 2 \cdot 23 + 13 \\
23 &= 1 \cdot 13 + 10 \\
13 &= 1 \cdot 10 + 3 \\
10 &= 3 \cdot 3 + 1 \\
3 &= 3 \cdot 1 + 0
\end{aligned}$$

Thus $(227, 143) = 1$ by the Euclidean algorithm.

3. Problem 1.2.3

   Claim. *If $a|b$ and $b|c$, then $a|c$*

   Proof. Since $a|b$, there exists $q \in \mathbb{Z}$ such that $b = q \cdot a$. Similarly, since $b|c$, there exists $r \in \mathbb{Z}$ such that $c = r \cdot b$ Thus $c = r \cdot (q \cdot a) = (r \cdot q) \cdot a$ (by associativity) so $a|c$. □

4. Problem 1.2.4 (a)

   Claim. *If $a|b$ and $a|c$, then $a|(b + c)$*

   Proof. Since $a|b$ we can write $b = q \cdot a$, and since $a|c$ we can also write $c = r \cdot a$. Thus $b + c = q \cdot a + r \cdot a = (q + r) \cdot a$ and therefore $a|(c + d)$. □

5. Problem 1.2.7

   Prove of disprove: If $a|(b + c)$, then $a|b$ or $a|c$.

   Proof. The above statement is false and we disprove it by a counterexample (any other method of proof is probably not the best idea, and more work than necessary if at all possible): For example, $2|(1 + 5)$ but $2 \nmid 1$ and $2 \nmid 5$. □

6. Problem 1.2.14 Find the smallest positive integer in the given set:

   (a) $\{6u + 15v \mid u, v \in \mathbb{Z}\}$

   (b) $\{12r + 17s \mid r, s \in \mathbb{Z}\}$

   (a) We can use the Euclidean algorithm to determing $(6, 15) = 3$. Thus by Hungerford, theorem 1.3 we have that 3 is the smallest positive integer in this set.

   (b) As in part (a) we find that $(12, 17) = 1$, and therefore the smallest positive integer in this set is 1.

7. Problem 1.2.20

   Prove or disprove each of the following statements:

   (a) If $2 \nmid a$, then $4 \mid (a^2 + 1)$.

   (b) If $2 \nmid a$, then $8 \mid (a^2 + 1)$

Proof. Both of these statements are true. For (a) we see that if $2 \nmid a$ then a is odd, that is $a = 2n + 1$. Then

$$\begin{aligned}(a^2 + 1) &= (a + 1)(a - 1) \\ &= (2n + 2)(2n) \\ &= 4(n + 1)n\end{aligned}$$

Part (b) is only slightly more difficult. From part (a) we know that $(a^2 + 1) = 4(n+1)n$. The point here is to notice that one of any two consecutive integers must be even. Then either n or $(n + 1)$ is even in which case we get another factor of 2 to divide $(a^2 + 1)$. Since $8 = 4 \cdot 2$ we're done!

□