These notes are based on my Advancement to Candidacy Lecture Notes, given on Friday, May 27, 2016. They have been expanded since then.

My aim is to tackle some questions relating to the modular representation theory of symmetric groups.

Sections 1 through 3 give some of the established results on the subject. Section 1 describes the CDE triangle associated to the modular representation theory, and expands it with the Hecke algebra at a root of unity serving as an intermediate between characteristic 0 and characteristic $p$. Section 2 describes the results that were conjectured and proved in the 1990s. Section 3 explains why these theorems don’t give us the desired decomposition matrix.

Section 4 describes what the recently discovered Cyclotomic Quiver Hecke Algebras have to offer. Section 5 covers the questions that I hope to help address. Section 6 gives the integral version of the main theorem of section 2. Section 7 gives a Quantum version of the same theorem, via Cyclotomic Quiver Hecke algebras. Section 8 defines the relevant Quantum groups and Fock space representations.

Let $S_n$ denote the symmetric group on $n$ letters. The main objects of interest will be $kS_n$, for $k$ a field; in particular, the finite fields $k = \mathbb{F}_p$.

For $k$ a field and $A$ a $k$-algebra, let $A$-mod denote the category of finite-dimensional $A$-modules, and let $A$-proj denote the full sub-category of finite-dimensional projective $A$-modules.

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1. The CDE triangle (and rectangle)

1.1 CDE Triangle associated to the symmetric groups

We start with the CDE triangle associated to the characteristic $p$ representation theory of $S_n$:
Here $K_0$ denotes the Grothendieck groups associated to these categories; we describe these Grothendieck groups and the maps explicitly here:

1. $K_0(\mathbb{Q}\mathcal{S}_n\text{-mod})$ is a free abelian group on isomorphism classes of simple $\mathbb{Q}\mathcal{S}_n$ modules. These are enumerable as follows: To each partition $\lambda$ of $n$, we associate the Specht module $S^\lambda$ over $\mathbb{Z}$, which has a very explicit combinatorial construction [11]. We have

$$\{\text{isomorphism classes of irreducible } \mathbb{Q}\mathcal{S}_n\text{-modules}\} = \{S^\lambda \otimes \mathbb{Q} | \lambda \vdash n\}.$$

2. $K_0(\mathbb{F}_p\mathcal{S}_n\text{-mod})$ is a free abelian group on isomorphism classes of simple $\mathbb{F}_p\mathcal{S}_n$ modules, which we also know how to enumerate [4]: for $\lambda$ a $p$-regular partition of $n$, i.e., one which does not repeat any part $p$ or more times, the $\mathbb{F}_p\mathcal{S}_n$ module $S^\lambda \otimes \mathbb{F}_p$ has a unique simple quotient, which we denote by $D^\lambda$.

**Example** For $p = 3$, $n = 9$, $(6, 3)$ and $(4, 3, 2)$ are $3$-regular partitions, but $(2, 2, 2, 1, 1, 1)$, $(2, 2, 2, 1)$, and $(3, 3, 3)$ are not.

3. $K_0(\mathbb{F}_p\mathcal{S}_n\text{-proj})$ is a free abelian group on isomorphism classes of projective indecomposable $\mathbb{F}_p\mathcal{S}_n$ modules. These are enumerated, for each $M$ a simple $\mathbb{F}_p\mathcal{S}_n$ module, by their projective covers $P_M \twoheadrightarrow M$.

- $d(S^\lambda \otimes \mathbb{Q}) = [S^\lambda \otimes \mathbb{F}_p]$. This is referred to as the Decomposition map, and we denote the Decomposition matrix of this map in terms of the above bases by $D$. It is surjective. The matrix has the form

$$
\begin{pmatrix}
1 & 0 \\
1 & \ddots \\
\vdots & \ddots & 1 \\
* & \ddots & \ddots & 1
\end{pmatrix}
$$

when the partitions are arranged with $p$-regular partitions first, and otherwise in lexicographic order. [8]

- $c([P]) = [P]$ — this corresponds to breaking down each indecomposable projective module into its irreducible composition factors. This is one version of the Cartan map, or Cartan matrix $C$ in terms of the above bases.

- $e$ comes from lifting projective modules over $\mathbb{F}_p\mathcal{S}_n$ to projective modules over $\mathbb{Z}_p\mathcal{S}_n$, then extending coordinates to $\mathbb{Q}$. The matrices $E$ and $D$ in terms of the bases are transpose to one another.

**Remark** The general theory of the CDE triangle requires that we work over a complete discrete valuation ring in place of $\mathbb{Z}_p$, e.g., the $p$-adic integers. This is unnecessary in the case of the symmetric groups, because $\mathbb{Q}$ is already a splitting field for the symmetric groups.

It is an unsolved problem to give a “nice” description of the decomposition matrix $D$ here — there are certainly brute force algorithms, since, e.g., $\{S^\lambda \otimes \mathbb{F}_p | \lambda \vdash n\}$ is a finite collection of finite objects to study. Knowing the decomposition maps would, in particular, make it possible to compute the dimensions of $D^\lambda$, which is also an unsolved problem. See, e.g., the preface of [7]. The decomposition matrices have been computed, for $p = 2$, up to $n = 17$ [9].

The theory we will elucidate below is better equipped to study $e$, but since $E$ and $D$ are transpose to one another, the two problems are readily seen to be equivalent.
1.2 The Hecke Algebra (CDE Rectangle)

There is an algebra which can play an intermediate role between $\mathbb{Q}S_n$ and $\mathbb{F}_pS_n$. This is the Hecke algebra, denoted by $H_q(S_n)$, an algebra over the base ring $\mathbb{Z}[q^{\pm 1}]$. It is given by generators $T_1, \ldots, T_{n-1}$, with relations given by

$$(T_i - q)(T_i + 1) = 0,$$

$$T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},$$

and

$$T_iT_j = T_jT_i \text{ when } |i - j| > 1.$$ 

For a commutative $\mathbb{Z}[q^{\pm 1}]$-algebra $R$, i.e., a commutative ring $R$ with a choice of $q \in R^\times$, we denote by $RH_q(S_n) := H_q(S_n) \otimes_{\mathbb{Z}[q^{\pm 1}]} R$. This algebra satisfies the following properties:

- For $q \in \mathbb{C}^\times$, $q$ not a root of unity, $CH_q(S_n)$ is (non-canonically) isomorphic to $\mathbb{C}S_n$, and we can classify the simple representations here canonically, by Specht modules analogous to the $\mathbb{C}S_n$ case.

- $\mathbb{Z}H_1(S_n) = \mathbb{Z}S_n$ canonically.

- Let $k \in \mathbb{Z}_>^0$, and $\zeta$ be a primitive $(p^k)^{\text{th}}$ root of unity. Then $\mathbb{Z}[\zeta]H_\zeta(S_n) \to \mathbb{F}_pS_n$ canonically. (This has to do with the previous fact and the fact that $\zeta \mapsto 1$ in $\mathbb{Z}[\zeta] \to \mathbb{F}_p$.)

- For $\zeta$ a primitive $e^\text{th}$ root of unity, the irreducible representations of $\mathbb{Q}(\zeta)H_\zeta(S_n)$ are enumerated by $S^\lambda \to D^\lambda$, for $\lambda$ an $e$-regular partition (analogous to the irreducible representations of $\mathbb{F}_pS_n$).

Based on the above facts, it was suggested by Geck, James and Lusztig that we can start with a “generic parameter” $q$, obtain a decomposition matrix by specializing $q$ to a $(p^k)^{\text{th}}$ root of unity, and then a second decomposition matrix by reducing modulo $p$ as before, providing a factorization of the above decomposition matrix (and, of course, of its transpose $E$). And indeed, we can do this. Since $\mathbb{Q}(\zeta)H_\zeta(S_n)$ is not semi-simple, we get a “CDE rectangle” instead of a triangle, as seen below.

We fix $k \in \mathbb{Z}_>^0$, let $e = p^k$, and fix $\zeta$ a primitive $e^\text{th}$ root of unity. We will abbreviate $\mathbb{Q}(\zeta)H_\zeta(S_n)$ by $H_n(\zeta)$.

(Here the name conflict between $e$, the maps above, and $e = p^k$ the number is insignificant, as we will stop referring to the maps by name.)

2 Fock Space

2.1 Definition of Fock Space

Focusing on the left side, we have

$$K_0(\mathbb{F}_pS_n\text{-proj}) \xrightarrow{e} K_0(H_n(\zeta)\text{-proj}) \xrightarrow{d} K_0(\mathbb{Q}S_n\text{-mod})$$

where we suppress the names of these maps, $e$, hereafter.

We take a direct sum over all $n$:  

3
Note that changing $k$ will only change the middle term, so we get many spaces which fit in between the two spaces on the left and right.

By convention, we let $S_0$ be the trivial group and $H_0(\zeta)$ be the trivial $\mathbb{Q}(\zeta)$ algebra. The abelian group on the right is then free abelian on partitions of all $n$, or Young diagrams, including the “empty partition” $\emptyset \vdash 0$, separate from the unique partition of 1. We will complexify, so let $F$ be the $\mathbb{C}$-vector space with basis given by partitions; we refer to this space as a Fock space.

The Fock Space has basis given by the nodes of the branching graph:

To each box in a Young diagram, we associate its content: If the box is in column $x$ and row $y$, then its content is given by $x - y$.

**Example** For $\lambda = (4, 4, 2, 1)$, the contents are given by:

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
-1 & 0 & 1 & 2 \\
-2 & -1 & & \\
-3 & & & \\
\end{array}
\]

### 2.2 Lie Algebra

There is a Lie algebra $\mathfrak{sl}_e$ which acts on $F$. This action is faithful, so we will take the approach of using the action to define the Lie algebra.
Let \( i \in \mathbb{Z}/e\mathbb{Z} \). We now label boxes by their contents modulo \( e \). We define \( e_i, f_i \in \text{End}(\mathcal{F}) \) by

\[
f_i(\lambda) = \sum_{\mu \text{ is obtained from } \lambda \text{ by adding an } i\text{-node}} \mu,
\]

and

\[
e_i(\lambda) = \sum_{\mu \text{ is obtained from } \lambda \text{ by removing an } i\text{-node}} \mu.
\]

**Example** Let \( e = 4 \), \( \lambda = (4, 1) \). The contents of \( \lambda \) modulo \( e \) are as follows:

\[
\begin{array}{cccc}
0 & 1 & 2 & 3 \\
\end{array}
\]

from which we get

\[
e_3(\lambda) = \begin{array}{cccc}
\hline
& & & \\
& & & \\
\end{array} + \begin{array}{cccc}
& & & \\
& & & \\
\end{array} = (4) + (3, 1)
\]

and

\[e_0(\lambda) = e_1(\lambda) = e_2(\lambda) = 0.\]

The addable nodes have contents 2, 0, and 0, in order from left to right. We get (with contents labeled for clarity):

\[
f_0(\lambda) = \begin{array}{cccc}
0 & 1 & 2 & 3 \\
3 & 0 & & \\
\end{array} + \begin{array}{cccc}
0 & 1 & 2 & 3 \\
3 & & & \\
\end{array} = (4, 2) + (5, 1),
\]

\[
f_2(\lambda) = \begin{array}{cccc}
0 & 1 & 2 & 3 \\
3 & & & \\
\end{array} = (4, 1, 1),
\]

and

\[f_1(\lambda) = f_3(\lambda) = 0.\]

**Definition** Let \( \hat{\mathfrak{sl}}_e \subset \text{End}(\mathcal{F}) \) be the complex Lie algebra generated by all \( e_i \) and \( f_i \) for \( i \in \mathbb{Z}/e\mathbb{Z} \).

We denote by \( U_e := U(\hat{\mathfrak{sl}}_e) \) the universal enveloping algebra.

Note that \( e_i \) and \( f_i \) are both locally nilpotent operators on \( \mathcal{F} \), because each partition has only finitely many \( i \)-nodes that can be added/removed at once before requiring nodes of other content. We may also refer to \( U_e^+ \) as the subalgebra generated by \( f_i \) and \( U_e^- \) as the subalgebra generated by \( e_i \).

In this manner, \( \mathcal{F} \) becomes a module over \( U_e \). The submodule generated by \( \emptyset \) is denoted by \( L_e(\Lambda_0) = U_e\emptyset \) which also equals \( U_e^+\emptyset \) (this has to do with a triangular decomposition of \( U_e = U_e^+U_e^0U_e^- \), where \( U_e^0 \) is generated by \( h_i := [e_i, f_i] \) for \( i \in \mathbb{Z}/e\mathbb{Z} \)).

**Remark** The notation \( L_e(\Lambda_0) \) comes from a classification

\[
\{ \text{Simple integrable } \}_{U_e\text{-mod}} \xrightarrow{\sim} (\mathbb{Z}^{\geq 0})^e,
\]

where \( (1, 0, 0, \ldots, 0) = \Lambda_0 \in (\mathbb{Z}^{\geq 0})^e \) is a fundamental weight. This is the “simplest” representation of \( U_e \).

Observe that \( U_p \hookrightarrow U_e \).
Theorem 2.1 \cite{3} \cite{7}
There are functors
\[ f_i : \mathbb{F}_p S_n \text{-proj} \to \mathbb{F}_p S_{n+1} \text{-proj}, \]
\[ e_i : \mathbb{F}_p S_n \text{-proj} \to \mathbb{F}_p S_{n-1} \text{-proj}, \]
for \( i \in \mathbb{Z}/p\mathbb{Z} \), which induce an action of \( U_p \) on \( \mathbb{C} \otimes_{\mathbb{Z}} \bigoplus_{n=0}^{\infty} K_0(\mathbb{F}_p S_n \text{-proj}) \).

Similarly, there are functors
\[ f_i : H_n(\zeta) \text{-proj} \to H_{n+1}(\zeta) \text{-proj}, \]
\[ e_i : H_n(\zeta) \text{-proj} \to H_{n-1}(\zeta) \text{-proj}, \]
for \( i \in \mathbb{Z}/e\mathbb{Z} \), which induce an action of \( U_e \) on \( \mathbb{C} \otimes_{\mathbb{Z}} \bigoplus_{n=0}^{\infty} K_0(\mathbb{F}_p S_n \text{-proj}) \).

We have a commutative diagram:

\[
\begin{array}{ccc}
\mathbb{C} \otimes_{\mathbb{Z}} \bigoplus_{n=0}^{\infty} K_0(\mathbb{F}_p S_n \text{-proj}) & \rightarrow & \mathbb{C} \otimes_{\mathbb{Z}} \bigoplus_{n=0}^{\infty} K_0(H_n(\zeta) \text{-proj}) \\
\downarrow \sim & & \downarrow \sim \\
\mathbb{C} \otimes_{\mathbb{Z}} \bigoplus_{n=0}^{\infty} K_0(\mathbb{Q}S_n \text{-mod}) & \rightarrow & \mathbb{C} \otimes_{\mathbb{Z}} \bigoplus_{n=0}^{\infty} K_0(\mathbb{Q}S_n \text{-mod})
\end{array}
\]

Each map commutes with the appropriate \( U_p \) or \( U_e \) module structure.

Remark The idea behind the Fock space here comes from Quantum Field theory: The \( f_i \) may be thought of as creation operators, the \( e_i \) may be thought of as annihilation operators, and the empty partition forms the vacuum vector.

Remark There is an integral version of \( U_e \) which allows us to express the same theorem without extending scalars to \( \mathbb{C} \). This takes a little bit more work to define, which is why we have not done so in this section. The integral version of this theorem is stated in section 6.

This gives explicit descriptions of the Grothendieck groups and their embeddings. Observe that as a corollary, when \( k = 1 \) the first map is an isomorphism. We can also use this to observe when the embeddings are isomorphisms for a fixed \( n \) — the embedding
\[ \mathbb{C} \otimes_{\mathbb{Z}} K_0(H_n(\zeta) \text{-proj}) \to \mathbb{C} \otimes_{\mathbb{Z}} K_0(\mathbb{Q}S_n \text{-mod}) \]
is an isomorphism if and only if \( e > n \).

3 The Problem

Theorem 2.1 does not provide us with the decomposition matrix, because we don’t know what the basis of indecomposable projectives is under the identification \( L_p(\Lambda_0) \simeq \mathbb{C} \otimes_{\mathbb{Z}} \bigoplus_{n=0}^{\infty} K_0(\mathbb{F}_p S_n \text{-proj}) \). Thus the problem of determining the decomposition matrix is reduced to finding the indecomposable projectives in this picture. In the case of the Hecke algebras, this is known:

Theorem 3.1 (Ariki) \cite{7}

Under the middle isomorphism of Theorem 2.1, the indecomposable projectives in \( \bigoplus_{n=0}^{\infty} K_0(H_n(\zeta) \text{-proj}) \) are identified with the upper global crystal basis of \( L_e(\Lambda_0) \).
This theorem makes reference to the upper global crystal basis associated to the Quantum group analogue of $U_e$, denoted $U_q(\mathfrak{sl}_e)$.

**Remark** Inside the Grothendieck group $K_0(\mathbb{F}_p S_n \text{-proj})$, there is the cone of classes of modules — the non-negative integer combinations of indecomposable projectives. If we can identify this cone then it is possible to discern what the set of indecomposable projectives is, namely they are the elements of the cone which cannot be decomposed inside the cone.

We have

$$ \left\{ \text{Cone generated by } \emptyset \text{ using } U_p^\mathbb{Z} \right\} \subset \bigcap_{k \geq 1} \bigg\{ \bigoplus_{n=0}^{\infty} K_0(\mathbb{F}_p S_n \text{-proj}) \bigg\} \cap \bigg\{ \bigoplus_{n=0}^{\infty} K_0(H_n(\zeta) \text{-proj}) \bigg\}, $$

which helps narrow down this cone. The description of the left hand side is explicit (and makes reference to the functors that arise in the main theorem of section 6), and each term of the right hand side is explicit — note that each value of $k$ corresponds to a distinct value of $\zeta$. Computing the intersection of these cones, however, is cumbersome.

This was the status quo around the mid to late 1990s [8] [1] [3]. The proofs of theorems 2.1 and 3.1 required the use of some of the deepest twentieth century mathematics, including the proof of the Weil conjectures.

When $n < p^2$, there is only one relevant cone on the right, namely $k = 1$ (the rest are trivial, in the sense that they are as large as is possible). The James conjecture [5] stated that in this circumstance, for $k = 1$, the decomposition matrix for $H_n(\zeta)$ over $\mathbb{F}_p S_n$ is the identity matrix — in other words, the inclusion of cones for this case is an equality. This conjecture was disproved three years ago [12].

### 4 Cyclotomic Quiver Hecke Algebras

A hint of the Quantum group of $U_e$ entered the picture in Ariki’s theorem above. In 2008, Rouquier [10], Khovanov, and Lauda [6] introduced a family of algebras which categorify certain Quantum groups. This family of algebras is referred to as the “Quiver Hecke Algebras”. In particular, we have the Cyclotomic quiver hecke algebras, which we will denote by $H_n(Q_e, \Lambda_0)$. This is a $\mathbb{Z}$-algebra, free abelian as an additive group with rank $n!$, which can be defined by (rather complicated) generators and relations.

One can understand $H_n(Q_e, \Lambda_0)$ as obtained by the following procedure: take a large commutative subalgebra in the Hecke algebra, divide up the algebra according to weight-spaces for the subalgebra, and find generators compatible with these weight spaces.

We have the following nice facts about this algebra:

- $H_n(Q_p, \Lambda_0) \otimes \mathbb{Z} \mathbb{F}_p \simeq \mathbb{F}_p S_n$,
- $H_n(Q_e, \Lambda_0) \otimes \mathbb{Z} \mathbb{Q}(\zeta) \simeq H_n(\zeta)$.
- $H_n(Q_e, \Lambda_0)$ comes equipped with a system of idempotents, making it easier to describe some projective modules. By contrast, group algebras defined over $\mathbb{Z}$ never have nontrivial idempotents.
- Most importantly: the cyclotomic quiver hecke algebras come equipped with a natural grading.

This grading enables one to make sense of the Quantum group associated to $U_e$ acting on the direct sum of Grothendieck groups

$$ \bigoplus_{n=0}^{\infty} K_0(kH_n(Q_e, \Lambda_0) \text{-graded-mod}), $$

where $k$ is any field. This is a $\mathbb{Z}[q^{\pm 1}]$-module, where $q$ acts by shifting the grading. Similar to Theorem 2.1 this sum of Grothendieck groups naturally identifies with the lowest weight representation $L_e(\Lambda_0)_q$ of $U_q(\hat{\mathfrak{sl}}_e)$ [2].
There are similar formulae for \(f_i, e_i\) defined on the Fock space \(\mathcal{F}_q := F \otimes \mathbb{C} \mathbb{C}(q)\), namely
\[
f_i(\lambda) = \sum_{\mu} q^{N_i^f(\lambda, \mu)} \mu,
\]
and
\[
e_i(\lambda) = \sum_{\mu} q^{-N_i^e(\mu, \lambda)} \mu
\]
where \(N_i^f(\lambda, \mu)\) and \(N_i^e(\mu, \lambda)\) are integers given in terms of the partitions.

5 Questions

1. Can we give an explicit combinatorial description of the right hand side cone seen above?

\[
\bigcap_{k \geq 1} \left\{ \bigoplus_{n=0}^{\infty} K_0(H_n(\zeta)-\text{proj}) \right\}
\]

2. What do the gradings of \(H_n(Q_p, \Lambda_0) \to \mathbb{F}_p S_n\) tell us about the cone?

3. What do the gradings of \(H_n(Q_e, \Lambda_0)\) tell us, for \(k > 1\)? The grading here induces a filtration on \(\mathbb{F}_p S_n\). Is this filtration compatible with the grading induced from the case \(k = 1\)?

4. In all of the above, \(\zeta\) has been chosen to be a fixed \(e^{th}\) root of unity. Changing the root of unity may change what happens in the representation theory of Hecke algebras. The Cyclotomic quiver hecke algebras provide a way of tying these together, in the following manner: let \(\zeta, \zeta'\) be distinct \(e^{th}\) roots of unity. Composing

\[
H_n(\zeta) \xrightarrow{\varphi} \mathbb{Q}(\zeta) H_n(Q_e, \Lambda_0) \xrightarrow{\varphi|_{\mathbb{Q}(\zeta)}} \mathbb{Q}(\zeta) H_n(\zeta', S_n) \xrightarrow{\varphi|_{\mathbb{Q}(\zeta)}} \mathbb{F}_p S_n
\]
gives a “natural” isomorphism between \(H_n(\zeta)\) and \(H_n(\zeta')\).
Theorem 6.1 (also proved in [3], [7])

Definition Let \( \ell \) for \( i \) and similarly

Remark Everything we have done here generalizes to higher level fock spaces \( L(\ell \cdot \Lambda_0) \), for \( \ell \geq 1 \), with partitions replaced by \( \ell \)-multipartitions. The combinatorics gets much more difficult here. [2]

6 Integral Version of Theorem 2.1

We now define an integral subalgebra of the \( U_c \), which we will denote by \( U_c^\mathbb{Z} \).

Definition Let \( \mathcal{F}_c^\mathbb{Z} \) denote the free abelian group on all partitions.

Observe that \( e_i \) and \( f_i \) act on \( \mathcal{F}_c^\mathbb{Z} \), for \( i \in \mathbb{Z}/e\mathbb{Z} \).

We define divided powers: For \( m \in \mathbb{Z}^>0 \) observe that \( (f_i)^m(\lambda) \) consists of all ways of adding \( m \) different \( i \)-nodes to \( \lambda \). Since the \( i \)-nodes are distance \( \geq 2 \) apart, they can be introduced in any order, which means that we overcount any particular choice of \( m \) \( i \)-nodes by precisely \( m! \), so we also have

\[
f_i^{(m)} := \frac{(f_i)^m}{m!} \in \text{End}(\mathcal{F}_c^\mathbb{Z}),
\]

and similarly

\[
e_i^{(m)} := \frac{(e_i)^m}{m!} \in \text{End}(\mathcal{F}_c^\mathbb{Z}).
\]

Definition Let \( U_c^\mathbb{Z} \) be the \( \mathbb{Z} \)-subalgebra of \( U_c \) generated by all \( e_i^{(m)} \) and \( f_i^{(m)} \), for \( i \in \mathbb{Z}/e\mathbb{Z} \) and \( m \in \mathbb{Z}^>0 \).

Let \( L_c(\Lambda_0)^\mathbb{Z} := U_c^\mathbb{Z} \oplus (U_c^\mathbb{Z})^+ \), where \( (U_c^\mathbb{Z})^+ \) is generated by all \( f_i^{(m)} \).

Theorem 6.1 (also proved in [3], [7])

There are functors

\[
f_i^{(m)} : \mathbb{F}_pS_n^\text{proj} \to \mathbb{F}_pS_{n+m}^\text{proj},
\]

\[
e_i^{(m)} : \mathbb{F}_pS_n^\text{proj} \to \mathbb{F}_pS_{n-m}^\text{proj},
\]

for \( i \in \mathbb{Z}/p\mathbb{Z} \) and \( m \in \mathbb{Z}^>0 \), which induce an action of \( U_c^\mathbb{Z} \) on \( \bigoplus_{n=0}^{\infty} K_0(\mathbb{F}_pS_n^\text{proj}) \).

Similarly, there are functors

\[
f_i^{(m)} : H_n(\xi)^\text{proj} \to H_{n+m}(\xi)^\text{proj},
\]

\[
e_i^{(m)} : H_n(\xi)^\text{proj} \to H_{n-m}(\xi)^\text{proj},
\]

for \( i \in \mathbb{Z}/e\mathbb{Z} \) and \( m \in \mathbb{Z}^>0 \), which induce an action of \( U_c^\mathbb{Z} \) on \( \bigoplus_{n=0}^{\infty} K_0(H_n(\xi)^\text{proj}) \).

The following diagram commutes and each map commutes with the appropriate \( U_c^\mathbb{Z} \) or \( U_c^\mathbb{Z} \) module structure:
7 Quantum Version of Theorem 2.1

I have diverted the first couple of definitions for this section to section 8.

The Quantum group $U_q(\mathfrak{sl}_c)$ is defined in section 8 as a $\mathbb{Q}(q)$-algebra. It has generators $E_i, F_i, K_i^{\pm 1}$ for $i \in \mathbb{Z}/c\mathbb{Z}$.

**Definition** Let $\mathcal{F}_q$ be the $\mathbb{Q}(q)$ vector space with basis given by partitions.

The Fock space representation of $U_q(\mathfrak{sl}_c)$ on $\mathcal{F}_q$ is given in section 8.

**Definition**

- Let $\mathcal{A} = \mathbb{Z}[q, q^{-1}]$ — this may be viewed as the quantum version of $\mathbb{Z}$ in section 6.

- Let $U_q^A(\mathfrak{sl}_c)$ be the $\mathcal{A}$-subalgebra of $U_q(\mathfrak{sl}_c)$ generated by $E_i^{(m)} := E_i^m / [m]!$, $F_i^{(m)} := F_i^m / [m]!$, and $K_i^{\pm 1}$.

  (Here the definition of $[m]!$ is given in section 8.)

  This is analogous to $U^A$. 

- Let $\mathcal{F}_q^A \subset \mathcal{F}_q$ be the free $\mathcal{A}$-module on partitions. Fact: the action of $U_q(\mathfrak{sl}_c)$ on $\mathcal{F}_q$ remains well defined when restricted to $U_q^A(\mathfrak{sl}_c)$ on $\mathcal{F}_q^A$.

  This is analogous to the fact that the action of $U_c$ on $\mathcal{F}$ restricts to a well-defined action of $U^Z$ on $\mathcal{F}^Z$.

- Let $L_c(\Lambda_0)_q^A$ be the $U_q^A(\mathfrak{sl}_c)$-submodule of $\mathcal{F}_q^A$ generated by $\emptyset$.

**Theorem 7.1** There are functors

$$F^{(m)}_i : \mathbb{F}_p H_n(Q_c, \Lambda_0)\text{-graded-proj} \to \mathbb{F}_p H_{n+m}(Q_c, \Lambda_0)\text{-graded-proj},$$

$$E^{(m)}_i : \mathbb{F}_p H_n(Q_c, \Lambda_0)\text{-graded-proj} \to \mathbb{F}_p H_{n-m}(Q_c, \Lambda_0)\text{-graded-proj},$$

for $i \in \mathbb{Z}/c\mathbb{Z}$ and $m \in \mathbb{Z}^{>0}$, which induce an action of $U_q^A(\mathfrak{sl}_c)$ on $ \bigoplus_{n=0}^{\infty} K_0(\mathbb{F}_p H_n(Q_c, \Lambda_0)\text{-graded-proj})$.

Similarly, there are functors

$$F^{(m)}_i : \mathbb{Q} H_n(Q_c, \Lambda_0)\text{-graded-proj} \to \mathbb{Q} H_{n+m}(Q_c, \Lambda_0)\text{-graded-proj},$$

$$E^{(m)}_i : \mathbb{Q} H_n(Q_c, \Lambda_0)\text{-graded-proj} \to \mathbb{Q} H_{n-m}(Q_c, \Lambda_0)\text{-graded-proj},$$

for $i \in \mathbb{Z}/c\mathbb{Z}$ and $m \in \mathbb{Z}^{>0}$, which induce an action of $U_q^A(\mathfrak{sl}_c)$ on $ \bigoplus_{n=0}^{\infty} K_0(\mathbb{Q} H_n(Q_c, \Lambda_0)\text{-graded-proj})$.

The following diagram commutes and each map commutes with the $U_q^A(\mathfrak{sl}_c)$-module structure:

$$\bigoplus_{n=0}^{\infty} K_0(\mathbb{F}_p H_n(Q_c, \Lambda_0)\text{-graded-proj}) \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} K_0(\mathbb{Q} H_n(Q_c, \Lambda_0)\text{-graded-proj})$$

$$\xrightarrow{\sim} \xrightarrow{\sim} L_c(\Lambda_0)_q^A \xrightarrow{\sim} L_c(\Lambda_0)_q^A$$

Note that we’ve lost the change from $p$ to $e$ in this version of the theorem — we don’t have a natural graded decomposition map from $\mathbb{Q} H_n(Q_c, \Lambda_0)$ down to $\mathbb{F}_p H_n(Q_p, \Lambda_0) \simeq \mathbb{F}_p S_n$ in general.
8 Definition of the Quantum Group and its Fock Space Representation

Let \( [a_{i,j}]_{i,j \in \mathbb{Z}/e\mathbb{Z}} \) be the \( e \times e \) matrix given by

\[
[a_{i,j}] = \begin{cases} 
\begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & 0 & -1 \\
-1 & 2 & -1 & \cdots & 0 & 0 & 0 \\
0 & -1 & 2 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & -1 & 0 \\
0 & 0 & 0 & \cdots & -1 & 2 & -1 \\
-1 & 0 & 0 & \cdots & 0 & 1 & 2 
\end{bmatrix} & \text{if } e > 2.
\end{cases}
\]

To help motivate the definition of \( U_q(\hat{\mathfrak{sl}}_e) \) by generators and relations, we first give the definition of \( U_e \) by generators and relations (in contrast to how we chose to define it in section 2). It is the \( \mathbb{C} \)-algebra given by generators \( e_i, f_i, h_i \) for \( i \in \mathbb{Z}/e\mathbb{Z} \), submitted to the following relations for all \( i, j \in \mathbb{Z}/e\mathbb{Z} \):

\[
[h_i, h_j] = 0, \quad [h_i, e_j] = a_{i,j} e_j, \quad [h_i, f_j] = -a_{i,j} f_j, \\
[e_i, f_j] = \delta_{i,j} h_i, \\
(ad e_i)^{1-a_{i,j}} e_j = 0, \quad (ad f_i)^{1-a_{i,j}} f_j = 0.
\]

Remark Actually, \( \hat{\mathfrak{sl}}_e \) is a little bit bigger than this — it is a central extension of this algebra. The same holds from the Quantum group defined below.

Now we're ready to define \( U_q(\hat{\mathfrak{sl}}_e) \). We start by defining the \( q \)-numbers, \( q \)-factorials, and \( q \)-binomial coefficients:

\[
[m] = q^m - q^{-m}, \\
[m]! = [m][m-1] \cdots [2][1], \\
\begin{bmatrix} m \\ k \end{bmatrix} = \frac{[m]!}{[m-k]![k]!}.
\]

Note that these are all Laurent polynomials.

We define \( U_q(\hat{\mathfrak{sl}}_e) \) as the \( \mathbb{Q}(q) \)-algebra generated by \( E_i, F_i, K_i^{\pm 1} \), for \( i \in \mathbb{Z}/e\mathbb{Z} \), together with relations for all \( i, j \in \mathbb{Z}/e\mathbb{Z} \):

\[
K_i K_j = K_j K_i, \quad K_i E_j K_i^{-1} = q^{a_{i,j}} e_j, \quad K_i F_j K_i^{-1} = q^{-a_{i,j}} F_j, \\
[E_i, F_j] = \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}.
\]
\[
\sum_{k=0}^{1-a_{i,j}} (-1)^k \binom{1}{k} E_i^{1-a_{i,j} - k} E_j E_k, \quad \sum_{k=0}^{1-a_{i,j}} (-1)^k \binom{1}{k} F_i^{1-a_{i,j} - k} F_j F_k.
\]

Now we define the Fock space representation of \( U_q(\hat{\mathfrak{sl}}_e) \) on \( \mathcal{F}_q \). Let \( \lambda, \mu \) be partitions so that \( \mu \) is obtained by adding an \( i \)-node \( \gamma \) to \( \lambda \).

Define

- \( R_i(\lambda) \) to be the number of \( i \)-removable nodes in \( \lambda \),
- \( I_i(\lambda) \) to be the number of \( i \)-addable nodes in \( \lambda \), and
- \( N_i(\lambda) = I_i(\lambda) - R_i(\lambda) \).

And relative versions:

- \( I_i^r(\lambda, \mu) \) is the number of \( i \)-addable nodes in \( \lambda \) to the right of \( \gamma \).
- \( R_i^r(\lambda, \mu) \) is the number of \( i \)-removable nodes in \( \lambda \) to the right of \( \gamma \).
- \( N_i^r(\lambda, \mu) = I_i^r(\lambda, \mu) - R_i^r(\lambda, \mu) \).
- \( I_i^l(\lambda, \mu) \) is the number of \( i \)-addable nodes in \( \lambda \) to the left of \( \gamma \).
- \( R_i^l(\lambda, \mu) \) is the number of \( i \)-removable nodes in \( \lambda \) to the left of \( \gamma \).
- \( N_i^l(\lambda, \mu) = I_i^l(\lambda, \mu) - R_i^l(\lambda, \mu) \).

Then \( U_q(\hat{\mathfrak{sl}}_e) \) acts on \( \mathcal{F}_q \) by

\[
F_i(\lambda) = \sum_{\mu} q^{N_i^r(\lambda, \mu)} \mu, \quad \text{\( \mu \) is obtained from \( \lambda \) by adding an \( i \)-node}
\]

\[
E_i(\lambda) = \sum_{\mu} q^{-N_i^l(\mu, \lambda)} \mu, \quad \text{\( \mu \) is obtained from \( \lambda \) by removing an \( i \)-node}
\]

and

\[
K_i \lambda = q^{N_i(\lambda)} \lambda.
\]

References


