REPRESENTATIONS OF SYMMETRIC GROUPS

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Abstract. A surprising theorem in the modular representation theory of symmetric groups uses induction and restriction functors to define an action of an affine Kac-Moody special linear algebra on the level of Grothendieck groups. This action identifies the direct sum of Grothendieck groups with an integrable highest weight module of the Kac-Moody algebra. The purpose of this write-up is to provide a gentle introduction to these theorems.

This write-up, still a work in progress, is mostly expository with few proofs. No technical background is assumed — all unfamiliar terms in this paper are either explicitly defined or blackboxed. The aim of this write-up is to be accessible to the average graduate student with enough patience. The reader who is rusty or unfamiliar with the basics of rings, fields, and modules may need to review those concepts first.

For the sake of presentation, a technical summary of the contents of this paper is included at the end instead of this introduction, in section 11.

You may have heard that the irreducible representations of symmetric groups over complex numbers can be classified explicitly by Young Diagrams. Sections 1 through 3 begin by describing this classification via eigenspace decompositions of certain elements of the group algebra, along with relevant induction and restriction functors. We then give some examples in section 4 of how one can use this theory to compute irreducible representations. Section 5 describes some relations satisfied by the induction and restriction functors.

We then elucidate in sections 6 through 8 how the same analysis generalizes to the characteristic $p$ case. In this case, the theory allows us to compute a large class of representations, but it is unknown how to compute the irreducible representations. A later version of this paper will probably include some example computations in this case, similar to section 4.

Sections 9 and 10 generalize the theory to related Cyclotomic Hecke algebras, which completes the picture of a major theorem which was conjectured by Lascoux, Leclerc, and Thibon and proved by Ariki in the 1990s.

This paper is a work in progress. Sections 1 through 7 are mostly completed.

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1. **Young Diagrams**

Let $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{Z}^k$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$.

**Definition 1.1.** The **Young Diagram** associated to $\lambda$ is the set of boxes in the left-adjusted figure consisting of $k$ rows with $\lambda_i$ boxes in the $i^{th}$ row, as depicted below.

**Example 1.2.** For $\lambda = (4, 4, 2, 1)$, the associated Young Diagram is given by:

```
  0 1 2 3
-1 0 1 2
-2 -1 2
-3
```

**Remark 1.3.** A Young Diagram can be defined rigorously as a subset of $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$, with pairs $(n, m)$ corresponding to boxes. In order to avoid getting bogged down with notation, we choose not to do so here.

**Definition 1.4.** Let $A$ be a box in the Young Diagram associated to $\lambda$, and assume it is in row $r$ and column $c$ of the diagram. Then the **content** of $A$ is $c - r$.

**Example 1.5.** For $\lambda = (4, 4, 2, 1)$, here are the contents of the boxes:

```
  0 1 2 3
-1 0 1 2
-2 -1 2
-3
```

Since Young Diagrams are in one-to-one correspondence with non-increasing sequences of positive integers, we choose to identify the two in our notation. For example, a box “in” $\lambda$ will refer to a box in its associated Young Diagram.

**Definition 1.6.** An $i$-addable box of $\lambda$ is a box $A$ not in (the Young Diagram of) $\lambda$, such that $A$ has content $i$ and $\lambda \cup \{A\}$ is a Young Diagram.

An $i$-removable box of $\lambda$ is a box $A$ in $\lambda$ with content $i$ such that $\lambda - \{A\}$ is a Young Diagram.

We have the branching graph for Young Diagrams below. Its vertices are given by Young Diagrams, and its edges are labeled by the content of the box which is added/removed. Note the inclusion of the empty diagram $\emptyset$, corresponding to $\lambda = ()$ and distinct from the diagram with one box corresponding to $\lambda = (1)$. 
We define the vector space $\mathcal{F}$, along with a set of operators on $\mathcal{F}$:

**Definition 1.7.** Let $\mathcal{F}$ be the complex vector space with (Hamel) basis given by Young Diagrams of all sizes. For $i \in \mathbb{Z}$, let $F_i$ and $E_i$ denote the linear operators on $\mathcal{F}$ given as follows on the basis:

$$F_i(\lambda) := \begin{cases} \lambda \cup \{A\} & \text{if } \lambda \text{ has an } i\text{-addable box } A, \\ 0 & \text{otherwise}. \end{cases}$$

$$E_i(\lambda) := \begin{cases} \lambda - \{A\} & \text{if } \lambda \text{ has an } i\text{-removable box } A, \\ 0 & \text{otherwise}. \end{cases}$$

The vertices of the branching graph form a basis for $\mathcal{F}$, and $F_i$’s follow $i$-edges up, and $E_i$’s follow $i$-edges going down.

**2. Symmetric Groups**

**Definition 2.1.** Let $F$ be a field, and $G$ a group. We define the **Group Algebra**, $F[G]$, to be the $F$-algebra with vector space basis given by the elements of $G$. Multiplication on the basis is given by the group operation, and extends to $F[G]$ by the distributive law.

Note that an $F$-algebra in this context is just a (unital, associative) ring $A$ together with a (unital) ring homomorphism $F \rightarrow A$ whose image is in the center of $A$. In the case of the Group Algebra, the image of $F \rightarrow F[G]$ is the one-dimensional vector space span $F \cdot 1$, where $1 \in G$ is the identity element.
Definition 2.2. Let $n \in \mathbb{Z}_{\geq 0}$. Let $S_n$ denote the Symmetric Group on $n$ letters, i.e., the group of permutations of the integer interval $[1, n] := \{1, 2, \ldots, n\}$.

Note that $S_0$ and $S_1$ are both trivial groups — the group of permutations on an empty set and a singleton, respectively. There is a canonical embedding $S_n \hookrightarrow S_{n+1}$ which sends $\sigma \in S_n$ to the permutation $\tau \in S_{n+1}$ such that $\tau_{[1,n]} = \sigma$ and $\tau(n+1) = n+1$.

Recall the cycle notation $(a_1 a_2 \cdots a_k)$ for the permutation which sends $a_i \mapsto a_{i+1}$ and $x \mapsto x$ when $x \notin \{a_1, \ldots, a_k\}$; this notation requires that the terms $a_1, \ldots, a_k$ be distinct. Every element of $S_n$ is a product of cycles, and this notation agrees with the natural embeddings $S_n \hookrightarrow S_{n+1}$.

We define all intervals in this write-up to be intervals of integers, as in Definition 2.2.

Definition 2.3. For $i \in [2, n]$, the Jucys-Murphy elements of $F[S_n]$ are $X_i := (1 \ i) + (2 \ i) + \cdots + (i-1 \ i) \in F[S_n]$. We set $X_1 := 0$.

Remark 2.4. The odd one out in this definition is obviously $X_1$. Its role in the theory will remain vacuous until section 9.

Remark 2.5. If $\sigma \in S_{i-1} \subset S_n$, then $\sigma X_i \sigma^{-1} = X_i$, because permuting $[1, i-1]$ only permutes the order of the terms defining $X_i$. So $X_i$ commutes with $\sigma$. Since $X_j$ for $j < i$ is a linear combination of elements of $S_{i-1}$, it follows that $X_i$ and $X_j$ commute.

Definition 2.6. For $G$ an arbitrary group, a Representation of $G$ over $F$ is defined to be a (left) module $M$ over $F[G]$ which is finite-dimensional as an $F$-vector space.

If $M$ is an arbitrary vector space over $F$, then a module structure on $M$ over $F[G]$ is the same data as that of a group homomorphism $S_n \rightarrow GL_F(M)$. This recovers the classical definition of a representation of a group.


More generally, for $A$ a unital associative $F$-algebra, $A$-Rep is defined to be the category of $F$-finite-dimensional modules over $A$. The notions of irreducibility, indecomposability, and semisimplicity can be defined on the level of modules:

Definition 2.7. For $R$ an arbitrary ring and $M$ an $R$-module, $M$ is said to be irreducible if $M$ has no $R$-submodules except $M$ and $\{0\}$. We say $M$ is indecomposable if for any isomorphism $M \cong N \oplus N'$ of $R$-modules, we have either $N$ or $N'$ is the trivial module $\{0\}$. We say $M$ is semisimple if it is a direct sum of irreducible modules.

We have the following theorem for representations of finite groups:

Theorem 2.8. (Maschke)

Let $G$ be a finite group and $F$ a field. Then every representation of $G$ over $F$ is semisimple if and only if the characteristic of $F$ does not divide the order of $G$.

Since all representations of a group in characteristic 0 are semisimple, it follows that, to classify representations of $G$ in characteristic 0, it is enough to classify their irreducible representations.

3. CLASSIFICATION OF IRREDUCIBLE REPRESENTATIONS IN CHARACTERISTIC 0

Now let’s fix $F = \mathbb{C}$. The goal of this section is to identify $\mathcal{F}$ above, together with the operations we defined there, in the representation theory of symmetric groups.
**Theorem 3.1.** Let $M \in \mathbb{C}[S_n]$-Rep be a complex representation of $S_n$. The Jucys-Murphy elements define diagonalizable linear maps on $M$, with eigenvalues in $\mathbb{Z}$.

Note that since $X_j$'s commute, it follows that the linear maps $X_1, \ldots, X_n$ can be simultaneously diagonalized on $M$. This fact motivates the following notation:

**Definition 3.2.** Let $I := \mathbb{Z} \subset \mathbb{C}$. For $i = (i_1, i_2, \ldots, i_n) \in I^n$, we denote by $M_i$ the simultaneous eigenspace of $(X_1, \ldots, X_n)$ with respective eigenvalues $i_1, \ldots, i_n$, that is,

$$M_i := \{ m \in M \mid X_j m = i_j m \ \forall j \in [1, n] \}.$$

It follows from the above that we have a direct sum decomposition of vector spaces

$$M = \bigoplus_{\alpha \in I^n} M_{\alpha}.$$

Note, however, that the $M_i$ are not subrepresentations. In order to get subrepresentations, we use the following theorem:

**Theorem 3.3.** The symmetric polynomials in $X_1, \ldots, X_n$ are contained in the center of $\mathbb{C}[S_n]$.

In fact, we have equality in the above theorem, but we don’t need that for the definitions below.

It is a standard fact that, as a consequence of them being central, it follows that simultaneous eigenspaces for the symmetric polynomials do form subrepresentations. Our next goal is to develop a notation for these subrepresentations. For the sake of clarity in exposition in defining this next piece of notation, we briefly work with the elementary symmetric polynomials in $X_j$,

$$Z_j := \sum_{s_1 < s_2 < \cdots < s_j \atop s_1, \ldots, s_j \in [1, n]} X_{s_1} X_{s_2} \cdots X_{s_j},$$

for $j \in [1, n]$, and consider simultaneous eigenspaces for $Z_j$.

Of course, $M_i$ forms a subspace of simultaneous eigenvectors for $Z_j$, for each $i \in I^n$. Since these subspaces span $M$, it follows that the simultaneous eigenspaces of $Z_j$'s will be sums of $M_i$'s.

Recall from the theory of symmetric polynomials that the eigenvalues of $Z_j$ acting on $M_i$ are precisely the coefficients of the polynomial

$$(x - i_1)(x - i_2) \cdots (x - i_n) \in \mathbb{C}[x]$$

up to an alternating sign. Furthermore, the data of these coefficients is the same as the data of the the roots of the polynomial with multiplicity, i.e., the data of the multiset $\{i_1, \ldots, i_n\}$.

This motivates the following definition:

**Definition 3.4.** Let $P$ be the free abelian group on formal symbols $\alpha_j$ for $j \in I$, i.e.,

$$P := \bigoplus_{j \in I} \mathbb{Z} \alpha_j.$$

For $\alpha \in P$, we define

$$M[\alpha] := \sum_{\alpha = \alpha_1 + \cdots + \alpha_n \atop \alpha \in I^n} M_{\alpha_1} \cdots M_{\alpha_n}.$$
In this notation, $\alpha$ serves the purpose of keeping track of the data of $\{i_1, \ldots, i_n\}$ as a multiset.

**Corollary 3.5.** $M[\alpha]$ is a subrepresentation of $M$ and
\[ M = \bigoplus_{\alpha \in P} M[\alpha]. \]

**Definition 3.6.** We refer to the $M[\alpha]$ as **blocks** of $M$.

There is one more piece of the puzzle for us to discuss before we get to the main theorems: induction and restriction. If $H \subset G$ are groups, then induction and restriction provide standard tools for relating the representations of $H$ to those of $G$. We define these over a general field.

**Definition 3.7.** Let $M \in \mathbb{F}[G]$-Rep. Then $\text{Res}_H^G(M) = M$ as a vector space, equipped with the $\mathbb{F}[H]$-module structure obtained by restricting the action of $\mathbb{F}[G]$ on $M$ to $\mathbb{F}[H]$.

Induction, unfortunately, has a more ad-hoc definition. We fix a set of left coset representatives $g_1, \ldots, g_k \in G$ for $H$, i.e., so that $G$ is expressed as a disjoint union of left cosets
\[ G = g_1H \sqcup \cdots \sqcup g_kH. \]

**Definition 3.8.** Let $M \in \mathbb{F}[H]$-Rep. We define
\[ \text{Ind}_H^G(M) := \bigoplus_{i \in [1,k]} g_iM, \]
where $g_i$ is a formal symbol. For arbitrary $g \in G$, $i \in [1,k]$, and $m \in M$, we define $g(g_i m)$ by rewriting $(gg_i)$ as $gg_i = g_jh$, for a unique $j \in [1,k]$ and $h \in H$, and setting $g(g_i m) := g_j(hm)$.

**Remark 3.9.** The ad-hoc definition can be avoided by defining induction as a tensor product over noncommutative rings $\text{Ind}_H^G(M) = \mathbb{F}[G] \otimes_{\mathbb{F}[H]} M$. Definition 3.8 then follows from the observation that $\mathbb{F}[G]$ forms a free right module over $\mathbb{F}[H]$. The choice of left coset representatives is a choice of a basis of this free module.

This also provides the definition of induction for arbitrary rings $A \subset B$.

We now have the tools to define the the class of “$i$-restricted” induction and restriction functors $F_i$ and $E_i$ on $\mathbb{C}[S_n]$-Rep, for each $i \in I$.

**Definition 3.10.** Fix $i \in I$, and let $M \in \mathbb{C}[S_n]$-Rep. We define the $i$-restricted restriction functor $E_i(M) \subset \text{Res}_{S_{n-1}}^{S_n} M$ to be eigenspace of $X_n$ with eigenvalue $i$. Recall from Remark 2.5 that $X_n$ commutes with $\mathbb{F}[S_{n-1}]$, so this eigenspace forms a subrepresentation of $\text{Res}_{S_{n-1}}^{S_n} M$.

It is a bit trickier to define $F_i$; we need to define it on the level of blocks. To motivate this approach, we provide the following alternative definition for $E_i$:

**Lemma 3.11.** For $\alpha \in P$, we have
\[ E_i(M[\alpha]) = (\text{Res}_{S_{n-1}}^{S_n} M[\alpha])[\alpha - \alpha_i], \]
and

\[ E_i(M) = \bigoplus_{\alpha \in P} E_i(M[\alpha]). \]

This should be pretty clear: If \( X_1, \ldots, X_{n-1} \) act on \( m \in M[\alpha] \) by the eigenvalues given in \( \alpha - \alpha_i \), then the only remaining eigenvalue for \( X_n \) on \( m \) is \( i \).

**Definition 3.12.** Fix \( i \in I \), and let \( M \in \mathbb{C}[S_n]\)\(-\text{Rep}\). The \( i \)-restricted induction functor \( F_i(M) \) is defined, for \( \alpha \in P \), by

\[ F_i(M[\alpha]) := (\text{Ind}_{S_n}^{S_{n+1}} M[\alpha])(\alpha + \alpha_i) \]

and

\[ F_i(M) = \bigoplus_{\alpha \in P} F_i(M[\alpha]). \]

Finally, we need to convert the representation theory of \( \mathbb{C}[S_n] \) into a vector space.

**Definition 3.13.** The Grothendieck Group \( \mathcal{K}_0(\mathbb{C}[S_n]) \) is the \( \mathbb{C} \)-vector space with basis given by \( [M] \) for each irreducible isomorphism class \( M \in \mathbb{C}[S_n]\)\(-\text{Rep}\).

Taking the direct sum of these Grothendieck groups over all \( n \), we now have the tools to state the theorems which classify the irreducible representations, and relate them to the Young Diagrams in the \( \mathcal{F} \). Abusing notation, we write \( E_i \) and \( F_i \) for both the linear operators on \( \mathcal{F} \) and the functors on representations of symmetric groups.

**Theorem 3.14.** The functors \( F_i \) and \( E_i \) induce operators on

\[ \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{K}_0(\mathbb{C}[S_n]). \]

There is a natural isomorphism of algebraic structures

\[ ( \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathcal{K}_0(\mathbb{C}[S_n]), \{E_i\}_{i \in I}, \{F_i\}_{i \in I} ) \]

with

\[ (\mathcal{F}, \{E_i\}_{i \in I}, \{F_i\}_{i \in I}). \]

(That is, there is a natural isomorphism of vector spaces which commutes with \( E_i \) and \( F_i \).)

Under this identification, irreducible representations correspond precisely with the basis of Young Diagrams in \( \mathcal{F} \).

Let \( \mathbb{C} \cdot 1 \) denote the trivial representation of \( S_0 \). This corresponds to \( \emptyset \in \mathcal{F} \). In order to abbreviate the statement of this next corollary, we let \( F_{\hat{i}} \) denote \( F_{i_n} \circ F_{i_{n-1}} \circ \cdots \circ F_{i_1} \) for \( \hat{i} = (i_1, \ldots, i_n) \in I^n \).

**Corollary 3.15.**

- Every irreducible representation of \( S_n \) is given by \( F_{\hat{i}}(\mathbb{C} \cdot 1) \) for some \( \hat{i} \in I^n \).
- For arbitrary \( \hat{i}, \hat{j} \in I^n \), we have \( F_{\hat{i}}(\mathbb{C} \cdot 1) \neq \{0\} \) if and only if \( F_{\hat{j}}(\emptyset) \neq 0 \), and the former defines an irreducible representation in this case.
- For an arbitrary pair \( \hat{i}, \hat{j} \in I^n \), we have \( F_{\hat{i}}(\mathbb{C} \cdot 1) \simeq F_{\hat{j}}(\mathbb{C} \cdot 1) \) if and only if \( F_{\hat{i}}(\emptyset) = F_{\hat{j}}(\emptyset) \) in \( \mathcal{F} \).
Moreover, for $M$ an irreducible $\mathbb{C}[S_n]$ representation, the eigenspace decomposition

$$M = \bigoplus_{i \in P} M_i$$

consists of 0 and 1-dimensional parts, with $\dim M_i = 1$ if and only if $M \cong F_i(\mathbb{C} \cdot 1)$, which in turn holds if and only if there is a path leading up from $\emptyset$ to the Young Diagram associated to $M$ with edges labeled by $i_1, i_2, \ldots, i_n$ in order.

The only part of this Corollary which doesn’t follow immediately from Theorem 3.14 is the last bullet point, which gives an explicit eigenspace decomposition. To see it as a consequence, observe that applying $E_i$’s in order singles out the eigenspaces, and restricting all the way down to $S_0$ allows you to identify their dimensions.

4. Examples

Theorem 3.15 provides an algorithm for determining the irreducible representations of $S_n$ inductively, albeit a kludgy one: Find the irreducible representations of $S_{n-1}$, and apply the functors $F_i$ to them. This involves taking the induced representations from $S_{n-1}$ to $S_n$ and computing simultaneous eigenspaces of the $X_i$’s on them, which is a straightforward linear algebra problem.

There do exist better methods for describing the irreducible representations of $S_n$. The purpose of this writeup is not to make computations easy, but rather to elucidate the theory from a particular perspective which generalizes to characteristic $p$. Nevertheless, we do include a few examples of computation here.

The first four examples will probably be familiar; we start with trivial representations and induction gives permutation representations. In particular, example 4.4 generalizes the first three examples. Example 4.5 has not yet been completed — it will elucidate the first “nontrivial” application of this approach.

Example 4.1. Let $V = \mathbb{C} \cdot v$ be the unique one-dimensional representation of $S_0$, and let $W = \text{Ind}_{S_0}^{S_1}$.

Since $S_0 = S_1$ as abstract groups, we have $W$ is also the trivial one-dimensional representation of $S_1$. This induction corresponds to adding the box of content 0 to the empty Young Diagram,

$$F_0(\emptyset) = \square$$

As such, identifying the subrepresentation of $W$ corresponding to $i = (0)$ corresponds to identifying the eigenspace of $X_1 = 0$ acting on $W$ with eigenvalue 0. All of this is rather vacuous.

Example 4.2. Let $V = \mathbb{C} \cdot v$ be the unique one-dimensional representation of $S_1$, and let $W = \text{Ind}_{S_0}^{S_1}$.

We saw above that $V$ corresponds to the Young diagram given by

$$\lambda := (1) = \square$$

This diagram has two addable boxes, with contents $-1$ and 1:

$$F_{-1}(\lambda) = \square$$

$$F_1(\lambda) = \square$$

But the Young diagrams given by $\lambda = F_{-1}(\lambda)$ and $\lambda = F_1(\lambda)$ are both still the box, and $X_{-1} V = 0$ and $X_1 V = 0$.
There is only one path in the branching graph from \( \emptyset \) to each of these diagrams. Thus we expect \( W \) to have two one-dimensional subrepresentations, given by

\[
W[\alpha_0 + \alpha_1] = W_{(0,1)}
\]

and

\[
W[\alpha_0 + \alpha_{-1}] = W_{(0,-1)}.
\]

The cosets of \( S_2 \) over \( S_1 \) are given by \{1\} and \{(1 2)\}, so \( W \) has a basis given by \( w_1 := v \) and \( w_2 := (1 2)v \). It is readily seen that \( (1 2)w_1 = w_2 \) and \( (1 2)w_2 = w_1 \).

It remains to identify the eigenspaces under the action \( X_2 = (1 2) \), with respective eigenvalues 1 and \(-1\):

\[
F_1(V) = W[\alpha_0 + \alpha_1] = W_{(0,1)} = \mathbb{C} \cdot (w_1 + w_2),
\]

\[
F_{-1}(V) = W[\alpha_0 + \alpha_{-1}] = W_{(0,-1)} = \mathbb{C} \cdot (w_1 - w_2).
\]

Note that \( F_1(V) \) is the trivial representation of \( S_2 \) and \( F_{-1}(V) \) is the sign representation.

**Example 4.3.** Let \( V = \mathbb{C} \cdot v \) be the one-dimensional trivial representation of \( S_2 \), given by \( gv = v \) for all \( g \in S_2 \). A good choice of left coset representatives of \( S_3 \) over \( S_2 \) is given by \( (1 3), (2 3), \) and \( 1 \).

We saw that the Young Diagram corresponding to \( V \) is given by

\[
\lambda := (2) = \begin{array}{|c|c|c|}
\hline
& & \\
\hline
& & \\
\hline
\end{array}
\]

Let \( W = \text{Ind}_{S_2}^{S_3} V \). Let \( w_1 := (1 3)v, w_2 := (2 3)v, \) and \( w_3 = v \). These \( w_i \)'s form a basis for \( W \), and one can compute the action of \( (i 3) \) on \( w_j \), for each \( i = 1, 2 \):

- If \( j = 3 \), then \((i 3)w_3 = (i 3)v = w_i.\)
- If \( j = i \), then \((i 3)w_i = (i 3)(i 3)v = v = w_3.\)
- The remaining cases consist of \( \{i, j\} = \{1, 2\} \):

\[
(1 3)w_2 = (1 3)(2 3)v = (1 3 2)v = (3 2 1)v = (3 2)(2 1)v = (2 3)v = w_2,
\]

and

\[
(2 3)w_1 = (2 3)(1 3)v = (2 3 1)v = (3 1 2)v = (3 1)(1 2)v = (1 3)v = w_1.
\]
Since (1 3) and (2 3) generate $S_3$, it follows that $W$ is the so-called permutation representation, or defining representation of $S_3$, meaning that $S_3$ acts directly on the basis of $W$ by permutations.

We wish to identify the simultaneous eigenspaces of $X_i$ on $W$. It is helpful to use the theory to tell use what to expect: $\lambda$ has two addable boxes, one with content 2, and one with content $-1$. We have

$$F_2(\lambda) = \begin{array}{c} \hline \hline \\ \end{array}$$

$$F_{-1}(\lambda) = \begin{array}{c} \hline \\ \hline \\ \end{array}.$$

So we expect two irreducible subrepresentations of $W$. There is only one path in the branching graph of Young Diagrams leading up to $F_2(\lambda)$, which corresponds to simultaneous eigenvalues $(0, 1, 2) \in \mathcal{I}^3$. It is easy to identify the corresponding subrepresentation:

$$W[\alpha_0 + \alpha_1 + \alpha_2] = W_{(0,1,2)} = \mathbb{C} \cdot (w_1 + w_2 + w_3).$$

The other irreducible subrepresentation is given by eigenvalues coming from the contents of the boxes in $F_{-1}(\lambda)$, which are given by $\alpha := \alpha_{-1} + \alpha_0 + \alpha_1$. The corresponding tuples of eigenvalues are given by the set of paths which lead up to $F_{-1}(\lambda)$ in the branching graph. There are two paths, given by contents/eigenvalues $(0, -1, 1)$ and $(0, 1, -1) \in \mathcal{I}^n$.

We observe that, according to the theory, $W_{(0, -1, 1)}$ should be the unique eigenspace of $X_2$ with eigenvalue $-1$, since $(0, -1, 1)$ is the only tuple we have with a $-1$ in the second position, and similarly $W_{(0,1,1)}$ is the eigenspace of $X_3$ with eigenvalue $-1$. This makes it relatively easy to find eigenvectors:

$$W_{(0,-1,1)} = \mathbb{C} \cdot (w_1 - w_2),$$

$$W_{(0,1,-1)} = \mathbb{C} \cdot (w_1 + w_2 - 2w_3),$$

and

$$W[\alpha] = W_{(0,-1,1)} \oplus W_{(0,1,-1)}.$$

This representation is often called the standard representation of $S_3$ — it is the 2-dimensional representation given by having $S_3$ permute the vertices of a triangle. To identify the vertices of a triangle in the above presentation, consider the images of $w_1, w_2, w_3$ in the projection map $W \twoheadrightarrow W[\alpha]$ given by the direct sum decomposition

$$W = \mathbb{C} \cdot (w_1 + w_2 + w_3) \oplus W[\alpha].$$

These images can be computed:

$$w_1 \mapsto (2/3)w_1 - (1/3)w_2 - (1/3)w_3,$$

$$w_2 \mapsto -(1/3)w_1 + (2/3)w_2 - (1/3)w_3,$$

and

$$w_3 \mapsto -(1/3)w_1 - (1/3)w_2 + (2/3)w_3.$$
Example 4.4. More generally, let $V = \mathbb{C} \cdot v$ be the one-dimensional trivial representation of $S_n$, given by $gv = v$ for all $g \in S_n$. A good choice of left coset representatives of $S_{n+1}$ over $S_n$ is given by $(1\ n+1), (2\ n+1), \ldots, (n-1\ n+1), (n\ n+1)$, and 1.

It will, of course, be helpful to identify the Young Diagram which corresponds to $V$. Since $(i\ j)$ acts by 1 for each $i \neq j$, $i, j \in [1, n]$, and $X_i$ consists of a sum of $i-1$ transpositions, it follows that $X_i$ acts by $i-1$ on $V$. So the unique eigenspace of $X_i$ on $V$ is given by simultaneous eigenvalues $(0, 1, 2, \ldots, n-1) \in \mathbb{F}_n$, with corresponding Young Diagram given by $\lambda = (n)$. For $n = 7$, for example:

$$\lambda = 
\begin{array}{cccccccc}
\text{•} & \text{•} & \text{•} & \text{•} & \text{•} & \text{•} & \text{•} \\
\end{array}
$$

Let $W = \text{Ind}_{S_n}^{S_{n+1}} V$. Let $w_i := (i\ n+1)v$ for $i = 1, 2, \ldots, n$ and $w_{n+1} := v$. These $w_i$’s form a basis for $W$, and one can compute the action of $(i\ n+1)$ on $w_j$, for each $i \leq n$:

- If $j = n+1$, then $(i\ n+1)w_{n+1} = (i\ n+1)v = w_i$.
- If $j \leq n$ and $i = j$, then $(i\ n+1)w_i = (i\ n+1)(i\ n+1)v = v = w_{n+1}$.
- If $i \neq j$, both $\leq n$, then

$$
(i\ n+1)w_j = (i\ n+1)(j\ n+1)v = (i\ n+1)j\ n+1v = (n+1\ j\ i)v = (n+1\ j\ i)v = (j\ n+1)v = w_j.
$$

We observe that, writing $\sigma = (i\ n+1)$, we have $\sigma w_j = w_{\sigma(j)}$. Since the set of $(i\ n+1)$ generate $S_{n+1}$, it follows that $W$ is the permutation representation of $S_n$.

We wish to identify the simultaneous eigenspaces of $X_i$ on $W$. It is helpful to use the theory to tell us what to expect: $\lambda$ has two addable boxes, one with content $n$, and one with content $-1$. We have $F_n(\lambda) = (n+1)$ and $F_{-1}(\lambda) = (n, 1)$. For $n = 7$:

$$
F_n(\lambda) = 
\begin{array}{cccccccc}
\text{•} & \text{•} & \text{•} & \text{•} & \text{•} & \text{•} & \text{•} \\
\end{array}
$$

$$
F_{-1}(\lambda) = 
\begin{array}{cccccccc}
\text{•} & \text{•} & \text{•} & \text{•} & \text{•} & \text{•} & \text{•} \\
\end{array}
$$

So we expect two irreducible subrepresentations of $W$, one of them being the trivial representation in the next level up, and the other one being $n$-dimensional. The corresponding eigenvalues for the trivial representation come from what we saw for $S_n$, namely $(0, 1, 2, \ldots, n)$. It is easy to give an eigenvector with these eigenvalues:

$$W_{(0,1,2,\ldots,n)} = \mathbb{C} \cdot (w_1 + \cdots + w_n).$$

This forms the trivial subrepresentation, that is, $W[\alpha_0 + \alpha_1 + \cdots + \alpha_n]$ is given by the above one-dimensional vector space.

The other irreducible subrepresentation is given by eigenvalues coming from the contents of the boxes in $F_{-1}(\lambda)$, which are given by $\alpha := \alpha_{-1} + \alpha_0 + \alpha_1 + \cdots + \alpha_{n-1}$. The corresponding tuples of eigenvalues are given by the set of paths which lead up to $F_{-1}(\lambda)$ in the branching graph. We will denote these by:

$$\iota^1 := (0, -1, 1, 2, 3, \ldots, n-2, n-1),$$
\[ i^2 := (0, 1, -1, 2, 3, \ldots, n - 2, n - 1), \]
\[ i^3 := (0, 1, 2, -1, 3, \ldots, n - 2, n - 1), \]
\[ \vdots \]
\[ i^{n-1} := (0, 1, 2, 3, \ldots, n - 2, -1, n - 1), \]
\[ i^n := (0, 1, 2, 3, \ldots, n - 2, n - 1, -1). \]

That is, \( i^k \) is the element of \( I^n \) which corresponds to adding the box with content \(-1\) after adding the box with content \( k - 1 \). Note that \( X_{k+1} \) should act on \( W_{i^k} \) with eigenvalue \(-1\), and moreover, this should be the only eigenspace of \( X_{k+1} \) with eigenvalue \(-1\). This observation makes it relatively straightforward to find an eigenvector:

\[ W_{i^k} = \mathbb{C} \cdot (w_1 + w_2 + \cdots + w_k - kw_{k+1}), \]

and

\[ W[\alpha] = W_{i^1} \oplus \cdots \oplus W_{i^n}. \]

This representation is often called the standard representation of \( S_{n+1} \) — it is the \( n \)-dimensional representation given by having \( S_{n+1} \) permute the vertices of an \((n + 1)\)-simplex. To identify the vertices of a simplex in this presentation, consider the images of \( w_1, w_2, \ldots, w_{n+1} \) in the projection map \( W \rightarrow W[\alpha] \) given by the direct sum decomposition

\[ W = \mathbb{C} \cdot (w_1 + \cdots + w_{n+1}) \oplus W[\alpha]. \]

These images can be computed, similarly to Example 4.3.

**Example 4.5.** Let \( V \) be the standard representation of \( S_3 \). A later version of this paper will include this example in full.

## 5. Relations

We now describe some of the relations that are satisfied by the operators \( E_i \) and \( F_i \) on Grothendieck groups. By Theorem 3.14, we can choose to focus instead on \( E_i \) and \( F_i \) as operators on \( \mathcal{F} \).

Note that for \( R \) a ring and \( a, b \in R \), we define the commutator bracket \([a, b] := ab - ba\). For reasons that will be made clearer in sections 9 and 10, we are only interested in relations which use the bracket.

Let \( H_i := [E_i, F_i] = E_i F_i - F_i E_i \). It is a straightforward combinatorial exercise to show that, for \( \lambda \) a Young Diagram,

\[ H_i(\lambda) = \begin{cases} 
\lambda & \text{if } \lambda \text{ has an } i\text{-addable box,} \\
-\lambda & \text{if } \lambda \text{ has an } i\text{-removable box,} \\
0 & \text{otherwise.}
\end{cases} \]

Let \( \{a_{i,j}\}_{i,j \in I} \) denote the following (infinite) matrix of integers:

\[ a_{i,j} := \begin{cases} 
2 & \text{if } i = j, \\
-1 & \text{if } i = j \pm 1, \\
0 & \text{otherwise.}
\end{cases} \]

The following theorem is also a straightforward combinatorial exercise. A later version of this paper will probably elucidate a proof.
Theorem 5.1. The operators $E_i, F_i, H_i$, for $i \in I$, satisfy the relations

\[
[h_i, h_j] = 0, \quad [h_i, e_j] = a_{i,j}e_j, \quad [h_i, f_j] = -a_{i,j}f_j, \quad [e_i, f_j] = \delta_{i,j}h_i,
\]

\[
(ad e_i)^{1-a_{i,j}}e_j = 0, \quad (ad f_i)^{1-a_{i,j}}f_j = 0.
\]

This theorem uses the adjoint action:

Definition 5.2. For $R$ a ring and $a \in R$, $ad a$ is the additive abelian group homomorphism

\[
ad a : R \to R,
\]

given by

\[
b \mapsto [a,b].
\]

We denote by $U_\infty$ the $\mathbb{C}$-algebra given by generators $E_i, F_i, H_i$ and relations above. A later version of this paper will describe how one can use the theory of integrable highest weight modules of $U_\infty$ to guide a proof of Theorem 3.14.

For now, as a preview of what such a description will include: notice that that $\mathcal{F}$ is an irreducible module over $U_\infty$: It is generated by applying $F_i$’s to $\emptyset$, and given $0 \neq v \in \mathcal{F}$ any $\mathbb{C}$-linear combination of Young Diagrams, $v$ can be reduced to a nonzero scalar multiple of $\emptyset$ by using $E_i$’s to remove boxes in order from a maximal size diagram in $v$.

To prove Theorem 3.14, it is sufficient to

- Prove that the operators $E_i, F_i, H_i := [E_i, F_i]$ acting on

\[
\bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{C}K_0(\mathbb{C}[S_n])
\]

satisfy the above relations.

- Prove that there is a surjective homomorphism of $U_\infty$-modules

\[
\mathcal{F} \twoheadrightarrow \bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{C}K_0(\mathbb{C}[S_n]).
\]

From this and the trivial observation that the direct sum of Grothendieck groups is not zero, it follows that the surjective homomorphism is an isomorphism.

6. REDEFINITIONS IN POSITIVE CHARACTERISTIC

Now we redefine everything for $F$ a field of characteristic $p$, $p > 0$. The theorems in this setting are surprisingly easy to state after we’ve done the hard work of stating their versions in characteristic 0. The difference is that Jucys-Murphy elements do not diagonalize, as they did in Theorem 3.1. However, they can be placed in upper triangular form:

Theorem 6.1. Let $M \in \mathbb{F}[S_n]$-Rep be a representation of $S_n$. The characteristic polynomials of $X_i$ acting on $M$ split in $F$, and the eigenvalues of $X_i$ all lie in the image of the unique ring homomorphism $\mathbb{Z} \to F$.

This motivates the change of notation from Definition 3.2:
Definition 6.2. Let \( I := \mathbb{Z}/p\mathbb{Z} \subset F \). For \( i = (i_1, i_2, \ldots, i_n) \in I^n \), we denote by \( M_i \) the simultaneous generalized eigenspace of \((X_1, \ldots, X_n)\) with respective eigenvalues \( i_1, \ldots, i_n \), that is,

\[
M_i := \{ m \in M \mid \forall j \in [1, n], (X_j - i_j)^N m = 0 \text{ for } N \text{ sufficiently large} \}.
\]

We have a direct sum decomposition of vector spaces

\[
M = \bigoplus_{i \in I^n} M_i,
\]

which are not subrepresentations. In order to get subrepresentations, we use the following theorem, same as Theorem 3.3:

**Theorem 6.3.** The symmetric polynomials in \( X_1, \ldots, X_n \) are contained in the center of \( F[S_n] \).

As before, we actually have equality in the above theorem, but we don’t need that for the definitions.

It follows that simultaneous generalized eigenspaces for the symmetric polynomials do form subrepresentations. As discussed in the characteristic 0 case, fixing a simultaneous generalized eigenspace for symmetric polynomials in \( X_i \) corresponds to fixing a choice of eigenvalues of \( X_1, \ldots, X_n \) up to permutation, so we define a notation which keeps track of the multiset of eigenvalues of \( X_1, \ldots, X_n \). With the above change in notation, Definition 3.4 is unchanged:

**Definition 6.4.** Let \( P \) be the free abelian group on formal symbols \( \alpha_j \) for \( j \in I \), i.e.,

\[
P := \bigoplus_{j \in I} \mathbb{Z} \alpha_j.
\]

For \( \alpha \in P \), we define

\[
M[\alpha] := \sum_{\alpha = \alpha_1 + \cdots + \alpha_n} M_{\alpha_i}.
\]

As before, \( \alpha \) serves the purpose of keeping track of the data of \( \{i_1, \ldots, i_n\} \) as a multiset.

**Corollary 6.5.** \( M[\alpha] \) is a subrepresentation of \( M \) and

\[
M = \bigoplus_{\alpha \in P} M[\alpha].
\]

**Definition 6.6.** We refer to the \( M[\alpha] \) as blocks of \( M \).

We don’t need to redefine induction and restriction, as they were already defined over general fields in section 3.

The class of “\( i \)-restricted” induction and restriction functors \( F_i \) and \( E_i \) on \( \mathbb{C}[S_n] \)-Rep have the same definitions for \( F[S_n] \), with generalized eigenspace in place of eigenspaces. In order to distinguish this setting from the characteristic 0 setting, we use \( e_i \) and \( f_i \) instead of \( E_i \) and \( F_i \).

**Definition 6.7.** Fix \( i \in I \), and let \( M \in F[S_n] \)-Rep. The \( i \)-restricted restriction functor \( e_i(M) \subset \text{Res}_{S_{n-1}}^{S_n} M \) is the generalized eigenspace of \( X_n \) with eigenvalue \( i \). By Remark 2.5, this generalized eigenspace forms a subrepresentation of \( \text{Res}_{S_{n-1}}^{S_n} M \).

We provide the following alternative definition for \( e_i \):
Lemma 6.8. For \( \alpha \in P \), we have
\[
e_i(M[\alpha]) = (\text{Res}^{S_n}_{S_{n-1}} M[\alpha])[\alpha - \alpha_i],
\]
and
\[
e_i(M) = \bigoplus_{\alpha \in P} e_i(M[\alpha]).
\]

Definition 6.9. Fix \( i \in I \), and let \( M \in \mathbb{C}[S_n]\text{-Rep} \). The \( i \)-restricted induction functor \( f_i(M) \) is defined, for \( \alpha \in P \), by
\[
f_i(M[\alpha]) := (\text{Ind}^{S_{n+1}}_{S_n} M[\alpha])[\alpha + \alpha_i]
\]
and
\[
f_i(M) = \bigoplus_{\alpha \in P} f_i(M[\alpha]).
\]

Finally, we need to convert the representation theory of \( F[S_n] \) into a vector space. Note that we still use Grothendieck groups with complex coefficients.

Definition 6.10. The Grothendieck Group \( \mathbb{C}K_0(F[S_n]) \) is the \( \mathbb{C} \)-vector space with basis given by \([M]\) for each irreducible isomorphism class \( M \in F[S_n]\text{-Rep} \).

As in the characteristic 0 case, the functors \( f_i \) and \( e_i \) induce operators on
\[
\bigoplus_{n \in \mathbb{Z}^\geq 0} \mathbb{C}K_0(F[S_n]).
\]

Finally, we revise our picture of \( F \). We keep the vector space \( F \) as in Definition 1.7.

Definition 6.11. Let \( i \in I \). We now label boxes by their contents modulo \( p \). We define \( e_i, f_i \) operators on \( F \) by
\[
f_i(\lambda) = \sum_{\substack{\mu \text{ is obtained} \\
\text{from } \lambda \text{ by} \\
\text{adding an } i\text{-box}}} \mu,
\]
and
\[
e_i(\lambda) = \sum_{\substack{\mu \text{ is obtained} \\
\text{from } \lambda \text{ by} \\
\text{removing an } i\text{-box}}} \mu.
\]

Note that we can relate these operators to those of Definition 1.7 by infinite sums which form finite sums on any fixed \( v \in F \):
\[
e_i = \sum_{n \in \mathbb{Z}} \left. E_i \right|_{n \equiv i \mod p},
\]
\[
f_i = \sum_{n \in \mathbb{Z}} \left. F_i \right|_{n \equiv i \mod p}
\]

The right analogue of \( F \) for the modular representation theory is a subspace of \( F \):

Definition 6.12. Let \( F_p \subset F \) be the smallest sub-vector-space which contains \( \emptyset \) and is closed under the operators \( e_i \) and \( f_i \).
7. Analogue of Theorem 3.14

Here is our first pass at an analogue of the Theorem:

**Theorem 7.1.** The algebraic structure

\[
\bigoplus_{n \in \mathbb{Z} \geq 0} \mathbb{C} K_0(F[S_n]), \{e_i\}_{i \in I}, \{f_i\}_{i \in I}
\]

is isomorphic to

\[(F_p, \{e_i\}_{i \in I}, \{f_i\}_{i \in I}).\]

However, that we don’t know how to identify the irreducible representations in this identification. Moreover this isomorphism is not very natural.

In order to get a natural isomorphism, we need to introduce a new technical tool from representation theory, which we treat as a black box:

**Definition 7.2.** Let \( R \) be a ring. A **projective module** over \( R \) is an \( R \)-module \( M \) such that there exists another (non-unique) \( R \)-module \( N \), a cardinal number \( c \) (possibly infinite), and an isomorphism

\[ M \oplus N \cong R^c \]

of \( M \oplus N \) with a free module of rank \( c \).

If \( A \) is an algebra over a field, then we denote by \( A\text{-Proj} \) the category of projective representations.

The “minimal” projective modules are the indecomposable projective modules, in the same sense that the minimal modules are the irreducible modules.

**Remark 7.3.** By Mashke’s theorem, the indecomposable projective representations over a group ring in characteristic zero are precisely the irreducible representations.

There is a very strong sense in which the problem of classifying irreducible representations is equivalent to the problem of classifying indecomposable projective representations. For example:

**Theorem 7.4.** Let \( A \) be a finite-dimensional algebra over a field \( F \) (e.g., a group algebra over a finite group). For every irreducible representation \( M \) of \( A \), there is an indecomposable projective representation \( P \) together with a surjective homomorphism

\[ P \twoheadrightarrow M, \]

and \( P \) is unique up to isomorphism. This \( P \) is called the **projective cover** of \( M \).

This assignment defines a bijection between isomorphism classes of irreducible representations and indecomposable projective representations.

We define the Grothendieck groups for projective representations.

**Definition 7.5.** The **Grothendieck Group** \( \mathbb{C} K_0(F[S_n]\text{-Proj}) \) is the \( \mathbb{C} \)-vector space with basis given by \([M]\) for each indecomposable isomorphism class \( M \in F[S_n]\text{-Proj} \).

The functors \( e_i \) and \( f_i \) send projective representations to projective representations.

**Theorem 7.6.** The functors \( f_i \) and \( e_i \) induce operators on

\[
\bigoplus_{n \in \mathbb{Z} \geq 0} \mathbb{C} K_0(F[S_n]\text{-Proj}).
\]
There is a natural isomorphism of algebraic structures

\[
\bigoplus_{n \in \mathbb{Z}_{\geq 0}} \mathbb{C}K_0(F[S_n] \text{-Proj}), \{e_i\}_{i \in I}, \{f_i\}_{i \in I}
\]

with

\[
(\mathcal{F}_p, \{e_i\}_{i \in I}, \{f_i\}_{i \in I}).
\]

(That is, there is a natural isomorphism of vector spaces which commutes with \(e_i\) and \(f_i\).)

**Remark 7.7.** The identification of indecomposable projective modules under this isomorphism is noticeably absent; this is an important unsolved problem in the modular representation theory of symmetric groups.

The sense in which this isomorphism is more natural is as follows:

**Theorem 7.8.** For each \(n\), there is a canonical embedding

\[
\mathbb{C}K_0(F[S_n] \text{-Proj}) \hookrightarrow \mathbb{C}K_0(\mathbb{C}[S_n] \text{-Rep})
\]

associated to the modular representation theory of groups.

We treat this embedding as a black box. For the reader interested in investigating further, this is the E part of the so-called CDE triangle of modular representation theory.

**Theorem 7.9.** The embedding of theorem 7.8 commutes with the embedding of \(\mathcal{F}_p\) into \(\mathcal{F}\). That is, the following diagram commutes:

\[
\begin{array}{ccc}
\bigoplus_{n=0}^{\infty} \mathbb{C}K_0(FS_n \text{-Proj}) & \xrightarrow{\sim} & \bigoplus_{n=0}^{\infty} \mathbb{C}K_0(\mathbb{C}S_n \text{-Rep}) \\
\mathcal{F}_p & \xrightarrow{\sim} & \mathcal{F}
\end{array}
\]

**Remark 7.10.** One can take the abelian subgroup of \(\mathbb{C}K_0(FS_n \text{-Proj})\) generated by indecomposable projective representations, which we defined to be linearly independent. Although we cannot identify the indecomposable representations in \(\mathcal{F}_p\), there is a version of Theorems 7.6 and 7.9 which allows us to identify this integer lattice in \(\mathcal{F}_p\). Stating this theorem requires defining divided power analogues of \(e_i\) and \(f_i\), which is why we have chosen not to do so here.

8. Relations for positive characteristic

9. (Degenerate) Cyclotomic Hecke algebras

From now on \(F\) is an arbitrary field. The notation is kept the same as in the previous sections, with differences depending on the characteristic of \(F\).

This section needs expanding, but in order to avoid leaving the reader hanging as to what the Cyclotomic Hecke algebras are, generators and relations are included.

**Definition 9.1.** The Affine Degenerate Hecke Algebra is the \(F\)-algebra given by generators \(s_1, \ldots, s_{n-1}\) and \(X_1, \ldots, X_n\) with relations

\[s_i^2 = 1,\]

\[s_is_j = s_js_i \text{ for } |i - j| > 1,\]

\[s_is_{i+1}s_i = s_{i+1}s_is_{i+1} \text{ for } 1 \leq i < n-1,\]

\[X_iX_j = X_jX_i \text{ for } |i - j| > 1,\]

\[X_iX_{i+1}X_i = X_{i+1}X_iX_{i+1} \text{ for } 1 \leq i < n-1.\]
\[ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}, \]

\[ s_i s_j = s_j s_i \text{ when } |i - j| > 1, \]

\[ X_i X_j = X_j X_i, \]

\[ s_i X_{i+1} = X_i s_i + 1, \]

\[ s_i X_j = X_j s_i \text{ if } j \notin \{i, i+1\}. \]

To define the Degenerate Cyclotomic Hecke algebras, we need to briefly review notation. We define \( I \) as above, as the image of the unique ring homomorphism \( \mathbb{Z} \to F \), i.e., \( I = \mathbb{Z}/p\mathbb{Z} \) where \( p \) is the characteristic of \( F \).

If \((a_i)_{i \in I}\) is a family of non-negative integers, all but finitely many of them equal to zero, then

**Definition 9.2.** The **Degenerate Cyclotomic Hecke Algebra** associated to \((a_i)\) is given by the quotient of the Affine Degenerate Hecke Algebra by the additional relation

\[
\prod_{i \in I} (X_1 - i)^{a_i}.
\]

As an important case, the symmetric group algebra is a Degenerate Cyclotomic Hecke Algebra with \( a_0 = 1 \) and \( a_i = 0 \) for \( i \neq 0 \).

10. **Cyclotomic Hecke Algebras**

As in the degenerate case, we have generators and relations for now, and a later version will expand on this.

Let \( q \in F^* \). It is typically assumed that \( q \neq 1 \). We establish a new notation

\[ I := \{q^i \mid i \in \mathbb{Z}\}. \]

**Definition 10.1.** The **Affine Hecke Algebra** is the \( F \)-algebra given by generators \( T_1, \ldots, T_{n-1} \) and \( X_1^{\pm 1}, \ldots, X_n^{\pm 1} \) and relations

\[
(T_i - q)(T_i + 1) = 0,
\]

\[
T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},
\]

\[
T_i T_j = T_j T_i \text{ when } |i - j| > 1,
\]

\[
X_i X_j = X_j X_i,
\]

\[
T_i X_i T_i = q X_{i+1},
\]

\[
T_i X_j = X_j T_i \text{ if } j \notin \{i, i+1\}. \]

If \((a_m)_{m \in I}\) is a family of non-negative integers, all but finitely many of them equal to zero, then

**Definition 10.2.** The **Degenerate Cyclotomic Hecke Algebra** associated to \((a_m)\) is given by the quotient of the Affine Degenerate Hecke Algebra by the additional relation

\[
\prod_{m \in I} (X_1 - m)^{a_m}. \]

The Hecke algebra is a Cyclotomic Hecke Algebra with \( a_1 = 1 \) and \( a_m = 0 \) for \( m \neq 1 \).
11. Summary

Sections 1 through 3 elucidate the analogue of this theorem in characteristic 0. Section 1 defines Young Diagrams and the associated Fock space $\mathcal{F}$, along with the relevant linear operators. Section 2 defines group algebras and a few important pieces of the symmetric group algebras over a general field. In particular, the Jucys-Murphy elements of the symmetric group algebras are defined.

Section 3 describes some of the nice properties of the Jucys-Murphy elements in characteristic 0 — in particular, we can use eigenspace decompositions of Jucys-Murphy elements to classify all irreducible representations. We accomplish this by way of certain induction and restriction functors. We will describe relations satisfied by these induction and restriction functors on the level of Grothendieck groups. We use these functors to realize an isomorphism between the direct sum of Grothendieck groups and the fock space $\mathcal{F}$ defined in section 1. Section 4 gives some sample computations to make the theorems more explicit.

Section 5 describes some relations satisfied by operators on the Fock space. These turn out to match those of a Lie algebra.

We then elucidate how this method of controlling the representations generalizes to the characteristic $p$ case. Section 6 gives the required redefinitions of notation and functors from the characteristic 0 case. Section 7 presents the major theorem, giving relations satisfied by the functors on the level of Grothendieck groups, which match those of another Lie algebra (more specifically, a Kac-Moody algebra). If we restrict to projective modules, then the Grothendieck groups in characteristic $p$ embed into those in characteristic 0, via the E part of the CDE triangle of modular representation theory. We see this embedding explicitly in terms of $\mathcal{F}$.

In section 9, we see how this generalizes to using the degenerate Cyclotomic Hecke algebras to categorify all irreducible integrable highest weight modules of the corresponding Lie algebras. This version of the categorification requires that we continue to consider the Cyclotomic Hecke algebras in characteristic $p$, as we did for symmetric groups. In section 10, we lift this categorification to characteristic 0 via the Cyclotomic Hecke algebras. In this characteristic 0 case, there is a theorem which classifies the irreducible representations, which we will be unable to state explicitly.

This completes the picture of a major theorem which was conjectured by Lascoux, Leclerc, and Thibon and proved by Ariki in the 1990s. The last unstated theorem utilizes the Quantum Group analogues of the relevant Kac-Moody algebras. This insight has paved the way for more recent developments — the Quiver Hecke algebras or KLR algebras, for which similar categorification theorems exist with the Quantum groups in place of the Lie algebras.