CLASSIFICATION OF FINITE-DIMENSIONAL SEMISIMPLE LIE ALGEBRAS

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Abstract. Every finite-dimensional Lie algebra is a semi-direct product of a solvable Lie algebra and a semisimple Lie algebra. Classifying the solvable Lie algebras is difficult, but the semisimple Lie algebras have a relatively easy classification. We discuss in some detail how the representation theory of the particular Lie algebra $\mathfrak{sl}_2$ tightly controls the structure of general semisimple Lie algebras, which enables their classification via root spaces, which we can see is a quite tractable problem.

We also discuss Lie correspondence connecting the theory of Lie algebras with that of Lie groups, which is where applications, e.g., in particle physics, tend to arise.

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1. Preliminaries

Definition 1.1. A (complex) Lie algebra is a $\mathbb{C}$-vector space $\mathfrak{g}$ together with a bilinear pairing

$$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

satisfying

1. $[a, b] = -[b, a]$ for all $a, b \in \mathfrak{g}$ (\iff $[a, a] = 0$ for all $a \in \mathfrak{g}$), (antisymmetry)
2. $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ for all $a, b, c \in \mathfrak{g}$ (Jacobi identity).

All Lie algebras will be complex.

Example 1.2. The prototypical example of a Lie algebra is: Given an associative $\mathbb{C}$-algebra $A$, let $[a, b]$ on $A$ be defined by $[a, b] = ab - ba$. We denote this resulting Lie algebra by Lie($A$).

In the particular example $A = \text{End}(V)$ for $V$ a vector space, we denote this by $\mathfrak{gl}(V) := \text{Lie(End}(V))$, and in particular, $\mathfrak{gl}_n = \mathfrak{gl}(\mathbb{C}^n)$.

Proof. Antisymmetry is obvious. The Jacobi identity follows from associativity as follows:

$$[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$$
\[ [a, [b, c]] + [b, [c, a]] + [c, [a, b]] \\
= [a, (bc - cb)] + [b, (ca - ac)] + [c, (ab - ba)] \\
= a(bc - cb) - (bc - cb)a \\
\quad + b(ca - ac) - (ca - ac)b \\
\quad + c(ab - ba) - (ab - ba)c \\
= abc - acb - bca + cba \\
\quad + bca - bac - cab +acb \\
\quad + cab - cba - abc + bac \\
= 0. \]

\[ \square \]

**Definition 1.3.** In light of example 1.2, it is natural to say that \( \mathfrak{g} \) is **abelian** if \([a, b] = 0\) for all \( a, b \in \mathfrak{g} \), and a fixed \( a \in \mathfrak{g} \) is **central** if \([a, b] = 0\) for all \( b \in \mathfrak{g} \).

**Definition 1.4.** If \( \mathfrak{g} \) is a Lie algebra, then we say a vector space \( \mathfrak{g}' \subset \mathfrak{g} \) is a **Lie subalgebra** if \([[-, -]] \) restricted to \( \mathfrak{g}' \times \mathfrak{g}' \) has image in \( \mathfrak{g}' \). We say \( \mathfrak{g}' \) is an **ideal** if \([[-, -]] \) restricted to \( \mathfrak{g} \times \mathfrak{g}' \) has image in \( \mathfrak{g}' \).

Note that every ideal is a subalgebra.

The Jacobi identity might appear somewhat mysterious; we saw that it is a consequence of associativity when \([[-, -]] \) is the commutator of an associative multiplication. The following theorem, which we won’t prove, can be seen as a sort of converse to this statement.

**Theorem 1.5.** (corollary of Poincaré-Birkhoff-Witt) For every Lie algebra \( \mathfrak{g} \), there is an associative algebra \( A \) such that \( \mathfrak{g} \) is isomorphic to a Lie subalgebra of \( \text{Lie}(A) \).

Thus, in some sense, the Jacobi identity can be seen as being “equivalent” to associativity.

**Definition 1.6.** A (Lie algebra) **homomorphism** \( f : \mathfrak{g} \to \mathfrak{g}' \) is a linear map with \( f([x, y]) = [f(x), f(y)] \) for all \( x, y \in \mathfrak{g} \).

**Definition 1.7.** A **representation** of a Lie algebra \( \mathfrak{g} \) is a Lie algebra homomorphism \( \mathfrak{g} \to \mathfrak{gl}(V) \) for some vector space \( V \).

That is, a representation \( \rho : \mathfrak{g} \to \mathfrak{gl}(V) \) is the data of \( \rho(x) : V \to V \), for each \( x \in \mathfrak{g} \), which varies linearly with respect to \( x \), with the property that \( \rho([x, y]) = \rho(x)\rho(y) - \rho(y)\rho(x) \) for every \( x, y \in \mathfrak{g} \). It will be common to suppress \( \rho \) when discussing representations; that is, we will write \( x : V \to V \), when we mean \( \rho(x) \). The notions of subrepresentation, quotient representation, irreducible representation, and direct sum of representations are defined in the usual manner.

**Example 1.8.** Every Lie algebra \( \mathfrak{g} \) is automatically a representation of itself, via the **adjoint action** where \( x \in \mathfrak{g} \) acts on \( \mathfrak{g} \) by \( \text{ad}(x) := [x, -] \).

Now assume that \( \mathfrak{g} \) is finite-dimensional.
Definition 1.9. We say that \( g \) is \textbf{simple} if it has no nontrivial ideals and is not abelian.

We say that \( g \) is \textbf{semisimple} if every finite-dimensional representation of \( g \) is a direct sum of irreducible representations.

We say an ideal is abelian if it is abelian as an algebra. We have the following somewhat surprising theorem, which we won’t prove:

**Theorem 1.10.** The following are equivalent:

1. \( g \) is semisimple,
2. \( g \) has no abelian ideal,
3. \( g \) is a direct sum of simple Lie algebras.

An algebraic proof that \( 2 \implies 1 \) might be compared to the proof of Maschke’s theorem, in that a bilinear form, called the \textbf{Killing form} is defined on an arbitrary representation of \( g \), which allows one to define complementary subrepresentations. This approach is a bit more subtle than the case for finite groups, and we have chosen not to discuss the Killing form in this writeup. This form is the source of many (though not all) of the black boxes here.

2. Special Linear Lie Algebra of Rank 2

**Definition 2.1.** We set \( \mathfrak{sl}_n \subset \mathfrak{gl}_n \) to be the set of matrices with trace zero.

Note that for any pair of matrices \( A, B \in \mathfrak{gl}_n \), \([A, B]\) has trace zero. It follows that \( \mathfrak{sl}_n \) is an ideal, and in particular a Lie subalgebra, of \( \mathfrak{gl}_n \).

We fix \( g \) := \( \mathfrak{sl}_2 \) in this section. It has basis given by

\[
e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

with relations

\[
[e, f] = h, [h, f] = -2f, [h, e] = 2e.
\]

Note that \( g \) is a simple Lie algebra: Let \( a \subset g \) be a nonzero ideal, and assume \( x = ae + bh + cf \in a \) is nonzero. Since \([h, -]\) diagonalizes on \( g \), each of the eigenspace projections \( ae, bh, cf \) can be expressed as a linear combination of \( x, [h, x] \), and \([h, [h, x]]\) and \( e, h, f \) each in turn generate \( g \) as an ideal.

Let \( V \) be a finite-dimensional representation of \( g \). That is, let \( \rho(e), \rho(f), \rho(h) \in \mathfrak{gl}(V) \) with

\[
\rho(e)\rho(f) - \rho(f)\rho(e) = \rho(h),
\]

\[
\rho(h)\rho(f) - \rho(f)\rho(h) = -2\rho(f),
\]

\[
\rho(h)\rho(e) - \rho(e)\rho(f) = 2\rho(e).
\]

We will suppress \( \rho \), in the notation.

To unravel the structure of \( V \), we first use the following theorem, which we black-box:

**Theorem 2.2.** \( h \) diagonalizes as an operator on \( V \).

**Definition 2.3.** An eigenvector for \( h \) with eigenvalue \( \lambda \) is said to have \textbf{weight} \( \lambda \). The eigenspace with eigenvalue \( \lambda \) is the \textbf{weight space} of \( V \) associated to \( \lambda \).
Let \( v \) be an eigenvector for \( h \), say with eigenvalue \( \lambda \). Then we have

\[
h(\text{ev}) = (he)v = (eh + 2e)v = e(h + 2)v = (\lambda + 2)v,
\]
and

\[
h(\text{fv}) = (hf)v = (fh - 2f)v = f(h - 2)v = (\lambda - 2)f v.
\]

**Remark 2.4.** Note that \( he \) and \( eh + 2e \) and \( h - 2 \) are not elements of \( \mathfrak{g} \), but they are well-defined operators on \( V \). In fact, they are elements of the universal enveloping algebra of \( \mathfrak{g} \), but we don’t need to discuss that in order to understand these expressions.

We can repeatedly iterate \( e \) to obtain eigenvectors with eigenvalue \( \lambda + 2, \lambda + 4, \lambda + 6 \), etc. Since eigenvectors with distinct eigenvalues are linearly independent, and \( V \) is assumed to be finite-dimensional, these vectors must eventually be zero.

**Definition 2.5.** A vector in \( v \in V \) is said to be **primitive**, or a **highest weight vector**, if it is an eigenvector for \( h \) and \( \text{ev} = 0 \). The scalar eigenvalue of \( h \) on \( v \) is called the **weight** of \( v \).

We may assume without loss of generality that \( v \) is primitive, by replacing \( v \) with \( e^nv \) for \( n \) such that \( e^n v \neq 0 \) and \( e^{n+1} v = 0 \).

Now we apply \( f \) to \( v \), where \( f^n v \) has eigenvalue \( \lambda - 2n \). Same as before, we must have \( f^n v = 0 \) for \( n \) sufficiently large.

However, there is an **obstruction** to \( f^n v \) being zero: If we apply \( e \) to \( f^n v \), then we get a scalar multiple of \( f^{n-1} v \):

- \( e(f^1 v) = (ef)v = (fe + h)v = f(ev) + hv = 0 + \lambda v = \lambda v, \)
- \( e(f^2 v) = (ef)(fv) = (fe + h)(fv) = f(efv) + hsv = \lambda fv + (\lambda - 2)(fv) = (2\lambda - 2)(fv), \)
- \( e(f^3 v) = (ef)(f^2 v) = (fe + h)(f^2 v) = f(ef^2 v) + hf^2 v = (2\lambda - 2)(f^2 v) + (\lambda - 4)(f^2 v) = (3\lambda - 6)(f^2 v), \)
- etc.

In general, this follows the pattern of the triangular numbers,

\[
\begin{align*}
\lambda f^{n+1} v &= (ef)(f^n v) = (fe + h)(f^n v) = f(ef^n v) + hf^n v \\
&= (n\lambda - n(n - 1))(f^n v) + (\lambda - 2n)(f^n v) = ((n + 1)\lambda - n(n + 1))(f^n v) \\
&= (n + 1)(\lambda - n)f^n v.
\end{align*}
\]

If \( f^{n+1} v = 0 \), then \( ef^{n+1} v \) cannot be a nonzero scalar multiple of \( f^n v \). So the natural question is, since we know that \( f^{n+1} v \) must be zero eventually: When does this obstruction break down?

The answer, of course, is when the scalar \( (n + 1)(\lambda - n) \) vanishes; in this case \( f^{n+1} v \) is allowed to be zero, and, in fact, must be zero in order for \( V \) to be finite-dimensional. Note that \( n \) here is a non-negative integer, so \( n + 1 \neq 0 \), hence \( \lambda = n \in \mathbb{Z}_{\geq 0} \). Recall that \( \lambda \) was chosen to be the weight of a primitive vector: A priori, it could have been any complex number, but we have discovered that it must be a non-negative integer!
Thus we have found that $Cv \bigoplus Cf v \bigoplus \cdots \bigoplus Cf^n v$ is a subrepresentation of $V$. Since the eigenspaces for $h$ are one-dimensional and they all generate the subrepresentation, it follows that this subrepresentation is irreducible. Denoting $v_i := f^i v$, we observe that $v_0, v_1, \ldots, v_{\lambda-1}, v_{\lambda}$ is a basis for the subrepresentation. We thus have the following theorem:

**Theorem 2.6.** For each non-negative integer $\lambda$, there is a unique irreducible representation of $\mathfrak{g}$ with highest weight $\lambda$, and these are all of the finite-dimensional irreducible representations of $\mathfrak{g}$. The action of $e, f, h$ on $V$ can be given as follows: $V$ has a basis $v_0, v_1, \ldots, v_{\lambda-1}, v_{\lambda}$ where $v_0$ is a primitive vector, and

- $fv_i = v_{i+1}, fv_{\lambda} = 0$,
- $ev_i = i(\lambda - i + 1)v_{i-1}, ev_0 = 0$,
- $hv_i = (\lambda - 2i)v_i$.

Since every representation is a direct sum of irreducible representations, this classifies all representations of $\mathfrak{g}$.

**Corollary 2.7.** If $V$ is an arbitrary finite-dimensional representation of $\mathfrak{g}$, then the weight spaces of $\lambda$ and $-\lambda$ have the same dimension.

In order to help motivate the notation in the next section, we describe the triangular decomposition of $\mathfrak{g}$: Recall that $f, h, e$ forms a basis for $\mathfrak{g}$. We define

- $\mathfrak{h} := \mathbb{C}h$,
- $\mathfrak{n}^+ := \mathbb{C}e$,
- $\mathfrak{n}^- := \mathbb{C}f$,

so that there is a vector space decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$, with the property that the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ diagonalizes.

Let $V$ be a representation of $\mathfrak{g}$. Since $h$ diagonalizes on $V$, it follows that $\mathfrak{h}$, which just consists of scalar multiples of $h$, “simultaneously” diagonalizes on $V$. The scalar action of $\mathfrak{h}$ on an eigenvector defines an element of the dual vector space $\mathfrak{h}^*$.

**Definition 2.8.** A simultaneous eigenvector for $\mathfrak{h}$ with eigenvalue $\lambda \in \mathfrak{h}^*$ is said to have weight $\lambda$. The eigenspace with simultaneous eigenvalues $\lambda$ is the weight space of $V$ associated to $\lambda$. The weights of $V$ are the set of elements of $\mathfrak{h}^*$ with nonzero weight space on $V$.

This definition will, of course, be more interesting when $\mathfrak{h}$ is not one-dimensional.

**Definition 2.9.** A nonzero weight of the adjoint action of $\mathfrak{h}$ on $\mathfrak{g}$ is called a root.

The roots of $\mathfrak{g}$ are given by $\{h \mapsto \pm 2\}$.

### 3. General Semisimple Lie Algebras

For this section, $\mathfrak{g}$ will be an arbitrary finite-dimensional semisimple Lie algebra. In the place of $\mathfrak{h}$ above, we have what is called a **Cartan subalgebra**.

**Theorem 3.1.** There is a maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ with the property that the adjoint action of each $h \in \mathfrak{h}$ on $\mathfrak{g}$ diagonalizes. This subalgebra is called a **Cartan subalgebra**; it is not unique, but it is unique up to automorphisms of $\mathfrak{g}$.
This is all we will need to know; the reader who is interested in understanding the proof of the existence of a Cartan subalgebra should note the definition of a Cartan subalgebra for an arbitrary finite-dimensional Lie algebra is more subtle, and the following is more precise statement for existence:

**Theorem 3.2.** For almost all \( x \in \mathfrak{g} \), the nil space of \( \text{ad}(x) \) acting on \( \mathfrak{g} \), that is, the generalized eigenspace of \( \text{ad}(x) \) with eigenvalue 0, defines a Cartan subalgebra.

Here, “almost all” means that the set of \( x \) for which this fails is the zero set of a polynomial function on \( \mathfrak{g} \) — it has measure zero, and is a Zariski closed subset of \( \mathfrak{g} \) considered as an algebraic variety.

Since \( \mathfrak{h} \) is abelian, it follows that for \( h, h' \in \mathfrak{h} \), \( \text{ad}(h) \) and \( \text{ad}(h') \) commute:

\[
(\text{ad}(h) \circ \text{ad}(h'))(x) = [h, [h', x]] = -[h', [x, h]] - [x, [h, h']] = [h', [h, x]] - 0 = (\text{ad}(h') \circ \text{ad}(h))(x).
\]

It therefore follows that the adjoint action of \( \mathfrak{h} \) on \( \mathfrak{g} \) simultaneously diagonalizes. If \( x \in \mathfrak{g} \) is a simultaneous eigenvector for \( \mathfrak{h} \), then let \( \lambda : \mathfrak{h} \to \mathbb{C} \) be the scalars such that \( [h, x] = \lambda(h)x \). It is obvious that \( \lambda \) varies linearly over \( \mathfrak{h} \), and therefore \( \lambda \in \mathfrak{h}^* \).

We say that \( x \) has **weight** \( \lambda \), and we denote by \( \mathfrak{g}^\lambda \) the set of all vectors with weight \( \lambda \), called the **weight space** with respect to the adjoint action of \( \mathfrak{h} \) on \( \mathfrak{g} \). If \( \lambda \neq 0 \), then we call \( \lambda \) a **root** of \( \mathfrak{g} \). We let \( R \subset \mathfrak{h}^* \) be the set of all roots.

Maximality of \( \mathfrak{h} \) as an abelian subalgebra implies \( \mathfrak{h} = \mathfrak{g}^0 \).

We require only one more black-box theorem, which comes directly from the Killing form:

**Theorem 3.3.** If \( \lambda \) is a root of \( \mathfrak{g} \), then \( [\mathfrak{g}^\lambda, \mathfrak{g}^{-\lambda}] = \{[x, y] \mid x \in \mathfrak{g}^\lambda, y \in \mathfrak{g}^{-\lambda}\} \) is one-dimensional, and \( \lambda \) restricted to \( [\mathfrak{g}^\lambda, \mathfrak{g}^{-\lambda}] \) is nonzero.

Implicit in this theorem is the claim that \( [\mathfrak{g}^\lambda, \mathfrak{g}^{-\lambda}] \subset \mathfrak{h} \), which follows from the first statement below. These are all the tools we need to prove that the structure of \( \mathfrak{g} \) is tightly controlled, in a few steps.

1. If \( x \in \mathfrak{g}^\lambda \) and \( y \in \mathfrak{g}^\mu \), then \( [x, y] \in \mathfrak{g}^{\lambda + \mu} \):

   For \( h \in \mathfrak{h} \), we have \( [h, [x, y]] = -[x, [y, h]] - [y, [h, x]] = -[x, -\mu y] - [y, \lambda x] = (\lambda + \mu)[x, y] \).

2. The set of roots \( \lambda \in R \) span \( \mathfrak{h}^* \):

   If it didn’t, then there would exist \( h \in \mathfrak{h} \) such that \( \lambda(h) = 0 \) for every root \( \lambda \). In other words, \( [h, x] = 0 \) for all \( x \in \mathfrak{g} \). But this implies \( h \) is central, and therefore spans an abelian ideal of \( \mathfrak{g} \), which is a contradiction.

3. For a fixed root \( \lambda \in R \), there is a unique element \( h_\lambda \) such that \( \lambda(h_\lambda) = 2 \).

   Fix nonzero elements \( e_\lambda \in \mathfrak{g}^\lambda \) and \( f_\lambda \in \mathfrak{g}^{-\lambda} \) such that \( [e_\lambda, f_\lambda] = h_\lambda \); such elements exist by theorem 3.3. We have

\[
[h_\lambda, e_\lambda] = \lambda(h_\lambda)e_\lambda = 2e_\lambda,
\]
\[
[h_\lambda, f_\lambda] = -\lambda(h_\lambda)f_\lambda = -2f_\lambda.
\]

Therefore \( f_\lambda, h_\lambda, e_\lambda \) span a 3-dimensional subalgebra of \( \mathfrak{g} \), isomorphic to \( \mathfrak{sl}_2 \).

This subalgebra is called an **\( \mathfrak{sl}_2 \) triple** of \( \mathfrak{g} \). We denote this subalgebra by \( \mathfrak{s}_\lambda \).
4. $\mathfrak{g}^\lambda$ and $\mathfrak{g}^{-\lambda}$ are one-dimensional:

Suppose there existed $f \in \mathfrak{g}^{-\lambda}$ which was not a scalar multiple of $f_\lambda$. Then by subtracting an appropriate scalar multiple of $f_\lambda$, we may conclude by Theorem 3.3 that $[e_\lambda,f] = 0$, and $[h,f] = -\lambda(h)f = -2f$. This makes $f$ a highest-weight vector for the adjoint action of $s_\lambda$ on $\mathfrak{g}$ with “weight” $-2$ in the sense of section 2, which contradicts Theorem 2.6.

The proof that $\mathfrak{g}^\lambda$ is one-dimensional is identical, with the roles of $e$ and $f$ interchanged.

5. If $\mu \in R$ is a root with $x \in \mathfrak{g}$ of weight $\mu$, then $[h_\mu,x] = \mu(h_\lambda)x$. This says that $x$ has “weight” $\mu(h_\lambda)$ under the adjoint action of $s_\lambda$ on $\mathfrak{g}$ in the sense of section 2.

It follows, for example, that $\mu(h_\lambda)$ is an integer.

6. If $\mu \in R$ is a root, then $\mu - \mu(h_\lambda)\lambda$ is a root:

Let $p = \mu(h_\lambda)$. Assume first that $p \geq 0$. Let $x \in \mathfrak{g}^\mu$ be nonzero. We observed that $x$ has weight $p$ in the sense of section 2. It follows from Theorem 2.6 that $x' := \text{ad}(f_\lambda)^p(x) \neq 0$, and from Step 1 that $x'$ has weight $\mu - p\lambda$.

The proof for $p \leq 0$ follows the same reasoning with $e_\lambda$ in place of $f_\lambda$.

7. Define the function $s_\lambda : \mathfrak{h}^* \to \mathfrak{h}^*$ by $s_\lambda(\phi) = \phi - \phi(h_\lambda)\lambda$. This is obviously linear in $\phi$, and by the previous step, $s_\lambda$ maps elements of $R$ to elements of $R$. We have

$$s_\lambda^2(\phi) = (\phi - \phi(h_\lambda)\lambda) - [\phi - \phi(h_\lambda)\lambda](h_\lambda)\lambda = \phi - \phi(h_\lambda)\lambda - \phi(h_\lambda)\lambda + \phi(h_\lambda)\lambda(h_\lambda)\lambda$$
$$= \phi - 2\phi(h_\lambda)\lambda + \phi(h_\lambda)(2)\lambda$$
$$= \phi.$$

So $s_\lambda^2$ is the identity. We also have $s(\lambda) = -\lambda$.

8. By Step 5, for $\mu \in R$, we have $s_\lambda(\mu) - \mu = -\mu(h_\lambda)\lambda$ an integer multiple of $\lambda$.

Steps 2, 7, and 8 form the definition of a root space:

**Definition 3.4.** Let $V$ be a vector space over a field $F$ with characteristic 0, and $R \subseteq V$ a finite set of nonzero vectors. This system is called a root space, and the elements of $R$ are called the roots of $V$, if

1. $R$ spans $V$,
2. For each $v \in R$, there exists a linear involution $s_v : V \to V$ which sends $v$ to $-v$, roots to roots, and fixes a hyperplane of $V$. (We will show that $s_v$ is unique, if it exists, as a consequence of condition (1).)
3. For $v,w \in R$, $s_v(w) - w$ is an integer multiple of $v$.

**Theorem 3.5.** If $R$ is a finite subset of a vector space $V$ over a field $F$ with characteristic 0, and $R$ spans $V$, then for $v \in V$, there is at most one linear involution $s : V \to V$ such that $s$ sends elements of $R$ to elements of $R$, $s(v) = -v$, and $s$ fixes a hyperplane of $V$.

**Proof.** Suppose $s, s'$ are two such linear maps. Then $ss'$ sends elements of $R$ to $R$ and $v$ to $v$. Moreover, both $s$ and $s'$ act by the identity on $V/Fv$. It follows that the eigenvalues of $ss'$ are all one, and since $R$ is finite and spans $V$, $(ss')^n = 1$ for some $n$ implies $ss'$ is diagonalizable. Therefore $ss' = 1$ and $s = s'$. □
Definition 3.6. A root system is **reduced** if for each root $\lambda$, $\pm \lambda$ are the only scalar multiples of $\lambda$ which are roots.

The following theorem isn’t too difficult to show based on the above work:

**Theorem 3.7.** The root system $(\mathfrak{h}^*, R)$ is reduced.

We have seen that the root system puts a fairly rigid structure on $\mathfrak{g}$. More precisely:

**Theorem 3.8.** For each complex reduced root system $(V, R)$, there is a unique semisimple Lie algebra whose associated root system $(\mathfrak{h}^*, R_\mathfrak{g})$ is isomorphic to $(V, R)$.

We haven’t actually demonstrated uniqueness; this involves selecting an element $X_\lambda \in \mathfrak{g}$ for each root $\lambda$, and then rescaling them so that the structure constants $[X_\lambda, X_\mu] = a_{\lambda,\mu} X_{\lambda+\mu}$ satisfy some nice properties (Chevalley’s normalization). Existence uses the Cartan matrix associated to the root system to produce a Lie algebra by generators and relations.

Finally, let’s state some properties of root systems. All proofs are omitted.

**Theorem 3.9.** Every complex root system is the complexification of exactly one real root system. More precisely, if $(V, R)$ is a complex root system, let $V_\mathbb{R}$ be the real span of $R$. Then we have $V = V_\mathbb{R} \otimes \mathbb{C}$.

So it is sufficient to understand real root spaces.

**Theorem 3.10.** Suppose $(V, R)$ is a real root system. There is a Euclidean inner product on $V$ which is invariant under the symmetries $s_\lambda$ for $\lambda \in R$.

This follows from the fact that the group generated by $\{s_\lambda\}_{\lambda \in R}$, called the **Weyl group**, is finite — one takes an arbitrary inner product and “averages” it over the Weyl group.

Let $\lambda, \mu \in R$, and let $\theta_{\lambda,\mu}$ be the angle between them with respect to the inner product. We’ll briefly consider $\lambda$ and $\mu$ to be fixed and $\theta = \theta_{\lambda,\mu}$. One can use condition (3) to show that $4 \cos^2 \theta$ is an integer. Moreover, if they are neither proportional nor orthogonal, then their relative lengths must be $2|\cos \theta|$. Thus, assuming without loss of generality that $|\mu| \geq |\lambda|$, the possibilities are:

- If $4 \cos^2 \theta = 4$, then $\lambda$ and $\mu$ are proportional, and $|\mu| \in \{|\lambda|, 2|\lambda|\}$ (this follows directly from condition (3)),
- If $4 \cos^2 \theta = 1$, then $\theta \in \{\pi/3, 2\pi/3\}$, and $|\mu| = |\lambda|$,  
- If $4 \cos^2 \theta = 2$, then $\theta \in \{\pi/4, 3\pi/4\}$, and $|\mu| = \sqrt{2}|\lambda|$,  
- If $4 \cos^2 \theta = 3$, then $\theta \in \{\pi/6, 5\pi/6\}$, and $|\mu| = \sqrt{3}|\lambda|$,  
- If $4 \cos^2 \theta = 0$, then $\theta = \pi/2$.

**Definition 3.11.** A **base** for a root system $(V, R)$ is a subset of $B \subset R$ which

- forms a basis for $V$, and
- For every root $\lambda \in R$, either $\lambda$ or $-\lambda$ is a non-negative integer linear combination of elements of $B$.

**Theorem 3.12.** Every root system has a base.

Note that the elements of a base are nonproportional, so the case $4 \cos^2 \theta = 4$ does not occur between distinct base elements.

**Definition 3.13.** The **Dynkin Diagram** associated to a (reduced) root system is a graph consisting of a vertex for each element of a base, along with 0, 1, 2, or 3 edges
joining the base elements $\lambda \neq \mu$ depending on whether $4 \cos^2 \theta_{\lambda,\mu} = 0, 1, 2,$ or $3$. In addition, if $4 \cos^2 \theta_{\lambda,\mu} = 2$ or $3$, then there is an arrow pointing in the direction of $\lambda$ if $|\mu| > |\lambda|$, and in the opposite direction otherwise (“the alligator wants to eat the bigger root”).

**Theorem 3.14.** Every root system is a direct sum of irreducible root systems. The reduced irreducible root systems are classified precisely by their Dynkin diagrams, as follows:

- $A_n$ for $n \geq 1$:

  ![Dynkin diagram for $A_n$]

- $B_n$ for $n \geq 2$:

  ![Dynkin diagram for $B_n$]

- $C_n$ for $n \geq 3$:

  ![Dynkin diagram for $C_n$]

- $D_n$ for $n \geq 4$:

  ![Dynkin diagram for $D_n$]

- $E_6$:

  ![Dynkin diagram for $E_6$]

- $E_7$:

  ![Dynkin diagram for $E_7$]

- $E_8$:

  ![Dynkin diagram for $E_8$]

- $F_4$:

  ![Dynkin diagram for $F_4$]

- $G_2$:

  ![Dynkin diagram for $G_2$]

Here the subscripts count the number of vertices, i.e., the size of the base and the dimension of the root space.
4. **Lie Correspondence**

**Definition 4.1.** A (complex) **Lie Group** is a complex manifold $G$ with analytic maps $m : G \times G \to G$ and $i : G \to G$ satisfying the usual axioms of group multiplication and inversion.

**Example 4.2.** For a finite-dimensional vector space $V$, $GL(V)$ defines a Lie group.

**Theorem 4.3.** Given a Lie group, there is a canonical Lie algebra associated to the tangent space at the identity. This defines a functor between categories

\[ \text{Lie} : \{ \text{Lie Groups} \} \to \{ \text{Lie Algebras} \}. \]

If $G$ is a connected Lie group, then its universal cover also has the canonical structure of a Lie group.

**Theorem 4.4.** (Lie correspondence)

The functor Lie restricted to simply connected Lie groups defines an equivalence of categories.

**Definition 4.5.** A **representation** of a Lie group $G$ is a homomorphism of Lie groups $G \to GL(V)$ for some finite-dimensional vector space $V$.

Note, however, that $GL(V)$ is not simply connected, so the above theorem doesn’t tell us that the representations of $G$ are the same as those of Lie($G$). In spite of this, we do have:

**Theorem 4.6.** Let $G$ be a simply connected Lie group, and $\mathfrak{g} = \text{Lie}(G)$. The functor Lie restricts to an equivalence of categories

\[ \text{Lie} : \{ \text{Representations of } G \} \to \{ \text{Representations of } \mathfrak{g} \}. \]

We make one more point: The above was for general lie groups with general Lie algebras (not necessarily semisimple). A Lie group $G$ is said to be **semisimple** if Lie($G$) is semisimple.

**Theorem 4.7.** Let $G$ be a semisimple Lie group. Then there is a faithful representation $G \to GL(V)$ for some $V$. Moreover, the image of this homomorphism is an algebraic subset of $GL(V)$, i.e., carved out by polynomial equations.

This allows one to assign to $G$ the structure of a (complex) **algebraic group** — a complex variety with a group structure of multiplication and inversion given by polynomial functions.

**Theorem 4.8.** Every complex analytic map between complex algebraic groups is an algebraic map.

This provides a dictionary between complex algebraic groups and Lie groups.

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