References: Vertex Operators and Algebraic Curves (second half may be chapter 5 instead of 4, depending on which edition you have), by Edward Frankel and David Ben-Zvi, and Infinite Dimensional Lie algebras (third edition) by Victor Kac.

1. Recall From Last Week

Definition 1. A vertex algebra is

- (space of states) a $\mathbb{Z} \geq 0$-graded vector space

$$V = \bigoplus_{m=0}^{\infty} V_m,$$

with $\dim V_m < \infty$,

- (vacuum vector) a vector $|0\rangle \in V_0$,

- (translation operator) a linear operator $T : V \to V$ of degree one,

- (vertex operators) a linear operation

$$Y(\cdot, z) : V \to \text{End} V[[z^{\pm 1}]]$$

which takes each $A \in V_m$ to a field

$$Y(A, z) = \sum_{n \in \mathbb{Z}} A(n) z^{-n-1}$$

(field, meaning that $\sum_{n \in \mathbb{Z}} A(n)(v) z^{-n-1}$ forms a formal Laurant series with coefficients in $V$, for each fixed $v \in V$) of conformal dimension $m$ (i.e., $\deg A(n) = (-n - 1) + m$).

which satisfy

- (vacuum axiom) $Y(|0\rangle, z) = \text{Id}_V$. For any $A \in V$,

$$Y(A, z)|0\rangle \in V[[z]],$$

and

$$Y(A, z)|0\rangle|z=0 = A.$$

- (translation axiom) For any $A \in V$,

$$[T, Y(A, z)] = \partial_z Y(A, z)$$

and $T|0\rangle = 0$.

All fields $Y(A, z)$ are pair-wise local with respect to each other, that is, $(z-w)^N$ annihilates $[Y(A, z), Y(B, w)]$ for $N \gg 0$ whenever $A, B \in V$.

Lemma 2. (Dong’s Lemma)

Let $A(z), B(z), C(z)$ be three mutually local fields. Then: $A(z)B(z) : is\ local\ with\ respect\ to \ C(z)$. 


Theorem 3. (The Reconstruction Theorem) Let

- $V$ be a $\mathbb{Z}_{\geq 0}$-graded vector space,
- $|0\rangle \in V_0$ nonzero,
- $T$ a degree 1 endomorphism of $V$,
- $S$ a countable ordered set,
- $\{a^\alpha\}_{\alpha \in S}$ a collection of homogeneous vectors in $V$, and
- $a^\alpha(z) = \sum_{n \in \mathbb{Z}} a^\alpha_{(n)} z^{-n-1}$ a collection of fields,

such that

1. For all $\alpha$, $a^\alpha(z)|0\rangle = a^\alpha + z(\cdots)$. (That is, the remaining nonzero terms are all positive degree in $z$.)
2. $T|0\rangle = 0$ and $[T, a^\alpha(z)] = \partial_z a^\alpha(z)$ for all $\alpha$.
3. All fields $a^\alpha(z)$ are mutually local.
4. $V$ has a basis of vectors $a_{(j_1)}^\alpha \cdots a_{(j_m)}^\alpha |0\rangle$,

for $j_1 \leq j_2 \leq \cdots \leq j_m < 0$ and $\alpha_i \leq \alpha_{i+1}$ whenever $j_i = j_{i+1}$.

Then the assignment

$$Y(a_{(j_1)}^\alpha \cdots a_{(j_m)}^\alpha |0\rangle, z) = \frac{1}{(-j_1 - 1)! \cdots (-j_m - 1)!} \partial_z^{-j_1 - 1 \alpha_1}(z) \cdots \partial_z^{-j_m - 1 \alpha_m}(z):$$

(given in terms of normally ordered products) defines a vertex algebra structure on $V$.

2. Affine Kac-Moody Algebras and Their Vertex Algebras


(§2.4.1 of F&B)

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra over $\mathbb{C}$. We define the formal loop algebra of $\mathfrak{g}$,

$$L\mathfrak{g} = g((t)) := \mathfrak{g} \otimes \mathbb{C}((t))$$

(or we could use $\mathbb{C}[t^{\pm 1}]$ in place of $\mathbb{C}((t))$). This comes equipped with the natural structure of a Lie algebra,

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t).$$

(This same definition works for any commutative $\mathbb{C}$-algebra.) We will define a Lie algebra $\widehat{\mathfrak{g}}$ as a central extension

$$0 \to \mathbb{C} K \to \widehat{\mathfrak{g}} \to L\mathfrak{g} \to 0,$$

where $K \in \widehat{\mathfrak{g}}$ is the central element. To define this, we use a $\mathfrak{g}$-invariant bilinear form $(\cdot, \cdot)$ on $\mathfrak{g}$, i.e.,

one satisfying $(\mathfrak{g}, \mathfrak{g}, z) + (y, [x, z]) = 0$ for all $x, y, z \in \mathfrak{g}$. It is well known that non-degenerate invariant forms on $\mathfrak{g}$ are unique up to rescaling. One such form is the Killing form,

$$(x, y)_K := \text{Tr}_\mathfrak{g}((\text{ad } x)(\text{ad } y)).$$

We will prefer to rescale by the following rule: Let $\mathfrak{h}$ be a Cartan subalgebra, and $\mathfrak{h}^*$ its dual. We define $(\cdot, \cdot)$ so that the square of the length of the maximal root of $\mathfrak{g}$, under the induced form on $\mathfrak{h}^*$, is equal to 2. It is known that

$$(\cdot, \cdot) = \frac{1}{2\mathfrak{h}^*}(\cdot, \cdot)_K,$$
where $h^\vee$ is the dual Coxeter number of $g$.

**Example 4.** If $g = sl_n$, then $(x, y) = Tr_{C^n}(xy)$ is the invariant form associated to the defining representation of $g$, rather than the adjoint representation.

Now we define $\hat{g}$ by the commutation relations

$$[K, \cdot] = 0,$$

$$[A \otimes f(t), B \otimes g(t)] = [A, B] \otimes f(t)g(t) - (\text{Res}_{t=0} f dg)(A, B)K.$$

That this formula is independent of the choice of coordinate $t$ will be important in later weeks.

Recall that $\hat{g}$ is a universal central extension of $Lg$, that is, every other central extension $\tilde{g}$ of $Lg$ admits a Lie algebra homomorphism $\hat{g} \to \tilde{g}$.

### 2.2. The vacuum representation.

(§2.4.2 of F&B)

Observe that $\hat{g}$ contains a Lie subalgebra $g[[t]] \oplus CK$ — the central extension becomes trivial here (as a consequence of the fact that residues vanish in the ring of formal power series).

Let $k \in \mathbb{C}$. We define a representation $C_k$ of $g[[t]] \oplus CK$ on which $g[[t]]$ acts by zero, and $K$ acts by scalar multiplication by $k$.

The *vacuum representation of level* $k$ of $\hat{g}$ is defined by

$$V_k(g) := \text{Ind}^{\hat{g}}_{g[[t]] \oplus CK} C_k = U(\hat{g}) \otimes_{U(g[[t]] \oplus CK)} C_k,$$

where induction of representations is defined as induction on the respective universal enveloping algebras.

More generally, for any module $M$ of $\hat{g}$, if $K$ acts by scalar multiplication by $k \in \mathbb{C}$, we say $M$ has level $k$.

The vector space decomposition

$$\hat{g} = (g \otimes t^{-1}C[t^{-1}]) \oplus (g[[t]] \oplus CK)$$

translates (by PBW) to an isomorphism of vector spaces

$$U(\hat{g}) \simeq U(g \otimes t^{-1}C[t^{-1}]) \otimes U(g[[t]] \oplus CK),$$

so

$$V_k(g) \simeq U(g \otimes t^{-1}C[t^{-1}]).$$

### 2.3. Vertex algebra structure.

(§2.4.4 of F&B)

Let $\{J^1, \ldots, J^{\dim g}\}$ be an ordered basis of $g$. For $A \in g$ and $n \in \mathbb{Z}$, denote

$$A_n := A \otimes t^n \in \hat{g}.$$

So $\{J^a_n \mid n \in \mathbb{Z}, 1 \leq a \leq \dim g\} \cup \{K\}$ forms a (topological) basis for $\hat{g}$, and the direct sum decomposition of $\hat{g}$ given above respects this basis. The commutation relations are

$$[J^a_n, J^b_m] = [J^a, J^b]_{n+m} + n(J^a, J^b)\delta_{n,-m}K.$$

Let $v_k$ denote the image of $1 \otimes 1$ in the tensor product definition of $V_k$. Since $g \otimes t^{-1}C[t^{-1}]$ has basis given by $J^a_n$ for $n < 0$, the Poincaré-Birkhoff-Witt theorem allows us to give a basis for $V_k(g)$; all we need is an ordering of the basis for $g \otimes t^{-1}C[t^{-1}]$. 


So we choose a Lexicographic ordering, yielding that \( V_k(g) \) has a algebraic basis given by monomials of the form

\[ J_{n_1}^{a_1} \cdots J_{n_m}^{a_m} v_k, \]

for \( n_1 \leq n_2 \leq \cdots \leq n_m < 0 \) and \( a_i \leq a_i + 1 \) whenever \( n_i = n_i + 1 \).

Now we give \( V_k(g) \) the structure of a (graded) vertex algebra:

- The gradation is given by \( \deg J_{n_1}^{a_1} \cdots J_{n_m}^{a_m} v_k = -\sum_{i=1}^{m} n_i \).
- Vacuum vector: \( |0\rangle = v_k \).
- Translation operator: \( Tv_k = 0 \), and \( [T, J_n^a] = -n J_{n-1}^a \).
- \( Y(v_k, z) = \text{Id} \in \text{End} V[[z^\pm 1]] \),

\[ Y(J_{-1}^a v_k, z) = J^a(z) = \sum_{n \in \mathbb{Z}} J_n^a z^{-n-1}, \]

and

\[ Y(J_{n_1}^{a_1} \cdots J_{n_m}^{a_m} v_k) \]

\[ = \frac{1}{(-n_1 - 1)! \cdots (-n_m - 1)!} : \partial_z^{-n_1-1} J_{a_1}^1(z) \cdots \partial_z^{-n_m-1} J_{a_m}^m(z) :. \]

**Theorem 5.** The above defines the structure of a vertex algebra on \( V_k(g) \).

**Proof.** We use the reconstruction theorem. This requires a generating set \( a^\alpha \), for which we take the vectors \( J_{-1}^a v_k \in V \) with associated fields \( J^a(z) \) as defined above.

1. We use the fact that \( J_n^a \) for \( n \geq 0 \) kills \( |0\rangle \). It follows immediately that \( J^a(z)|0\rangle = J_{-1}^a |0\rangle + z(\cdots) \), i.e., the remaining nonzero terms all have positive degree in \( z \).
2. \( [T, J_n^a(z)] = [T, \sum_{n \in \mathbb{Z}} J_n^a z^{-n-1}] \)
   
   \[ = \sum_{n \in \mathbb{Z}} [T, J_n^a] z^{-n-1} \]
   
   \[ = \sum_{n \in \mathbb{Z}} -n J_{-n-1}^a z^{-n-1} \]

   which agrees with \( \partial_z J^a(z) \) by a change of variables.
3. We simply calculate
\[
[J^a(z), J^b(w)] = \sum_{n,m \in \mathbb{Z}} [J^a_n, J^b_m] z^{-n-1} w^{-m-1}
\]
\[
= \sum_{n,m \in \mathbb{Z}} [J^a_n, J^b_m] z^{-n-1} w^{-m-1} + \sum_{n \in \mathbb{Z}} n(J^a, J^b) K z^{-n-1} w^n
\]
\[
= \sum_{l \in \mathbb{Z}} [J^a, J^b]_l \left( \sum_{n \in \mathbb{Z}} z^{-n-1} w^n \right) w^{-l-1} + (J^a, J^b) K \sum_{n \in \mathbb{Z}} n z^{-n-1} w^n
\]
\[
= \sum_{l \in \mathbb{Z}} [J^a, J^b]_l w^{-l-1} \left( \sum_{n \in \mathbb{Z}} z^{-n-1} w^n \right) + (J^a, J^b) K \sum_{n \in \mathbb{Z}} n z^{-n-1} w^n
\]
\[
= [J^a, J^b](w) \delta(z - w) + (J^a, J^b) k \partial w \delta(z - w).
\]

It follows that \((z - w)^2\) kills the commutator, so \(J^a(z)\) and \(J^b(w)\) are local with respect to one another.

(4) We have already stated this using the PBW basis.

\[\square\]

**Remark.** We have \(J^a(z) = \sum_n (J^a \otimes t^n) z^{-n-1}\); if we abuse notation a bit, this looks like \(J^a \otimes \delta(t - z)\). In this way, we may think of \(J^a(z)\) as an “operator-valued delta function” on \(V_k(g)\). To make this precise requires some analytic structure; we can’t really see the structure of an operator associated to every point in the disk on the algebraic side of things.

3. The Virasoro Vertex Algebra

3.1. The Virasoro algebra.

(§2.5.1 of F&B)

Let \(\mathcal{K} := \mathbb{C}(\!(t)\!)\) and \(\mathcal{O} := \mathbb{C}[\![t]\!]\). These notations will be fairly ubiquitous in weeks to come. As before, all definitions can be done with \(\mathbb{C}[\!(t^\pm 1)\!]\) and \(\mathbb{C}[t]\) instead, respectively.

Let \(\text{Der} \mathcal{K} := \mathbb{C}(\!(t)\!)[\partial_t]\) be the Lie algebra of continuous derivations of \(\mathcal{K}\). It is straightforward to compute that the Lie bracket is given by \([f \partial_t, g \partial_t] = (fg' - f'g) \partial_t\).

The Virasoro algebra is defined as a central extension

\[
0 \rightarrow \mathbb{C}C \rightarrow \text{Vir} \rightarrow \text{Der} \mathcal{K} \rightarrow 0
\]

given by

\[
[f(t) \partial_t, g(t) \partial_t] := (fg' - f'g) \partial_t - \frac{1}{12} (\text{Res}_{t=0} fg''' dt) C.
\]

This extension is universal, with topological basis \(C\) and \(L_n := -t^{n+1} \partial_t\) for \(n \in \mathbb{Z}\), on which the commutation relations are

\[
[L_n, L_m] = (n - m) L_{n+m} + \frac{n^3 - n}{12} \delta_{n,-m} C,
\]

where the last term comes from

\[
\text{Res}_{t=0} t^{n+1} (m + 1) m (m - 1) t^{m-2} dt.
\]
This is our first definition of a central extension which is not coordinate independent. A coordinate independent Virasoro algebra does exist, and we will see this in later weeks. However, there is no canonical splitting of the extension as a vector space, which is why the above definition depends on \( t \).

The relations on \( L_n \) can be summarized using the generating function

\[
T(z) := \sum_{n \in \mathbb{Z}} L_n z^{-n-2}.
\]

**Lemma 6.** We have

\[
[T(z), T(w)] = \frac{C}{12} \partial_w^3 \delta(z - w) + 2T(w) \partial_w \delta(z - w) + \partial_w T(w) \cdot \delta(z - w).
\]

**Proof.** We have

\[
[T(z), T(w)] = \sum_{n,m} (n - m)L_{n+m}z^{-n-2}w^{-m-2} + O \sum_n \frac{n^3 - n}{12} z^{-n-2}w^{n-2}.
\]

Substituting \( j = n + m \) and \( l = n + 1 \), we split \( n - m \) into \( 2l + (-j - 2) \), yielding

\[
= \sum_{j,l} 2lL_j w^{-j-2}z^{-l-1}w^{l-1} + \sum_{j,l} (-j - 2)L_j w^{-j-3}z^{-l-1}w^l + C \frac{1}{12} \sum_l l(l - 1)(l - 2)z^{-l-1}w^{l-3}
\]

\[
= 2T(w) \partial_w \delta(z - w) + \partial_w T(w) \cdot \delta(z - w) + \frac{C}{12} \partial_w^3 \delta(z - w).
\]

\[\square\]

### 3.2. The Virasoro Vertex Algebra.

(§2.5.6 of F&B)

We will define \( \text{Vir}_c \), following a very similar pattern.

Let \( \text{Der} \mathcal{O} = \mathbb{C}[[t]]\partial_t \), and observe that \( \text{Der} \mathcal{O} \oplus \mathbb{C}C \) is a Lie subalgebra of \( \text{Vir} \).

(\( \text{Der} \mathcal{O} \) is topologically generated by \( L_n \) for \( n \geq -1 \), so this simply comes from the fact that \( \frac{n^3 - n}{12} \delta_n \) vanishes for \( n, m \geq -1 \). The only tricky case is \( n = -m = \pm 1 \), but then \( n^3 - n \) vanishes.)

For \( c \in \mathbb{C} \), let \( \mathbb{C}_c \) be the representation of \( \text{Der} \mathcal{O} \oplus \mathbb{C}C \) on which \( \text{Der} \mathcal{O} \) acts by zero and \( C \) acts by multiplication by the scalar \( c \). Then we define

\[
\text{Vir}_c := \text{Ind}_{\text{Der} \mathcal{O} \oplus \mathbb{C}C}^{\text{Vir}_c} \mathbb{C}_c = U(\text{Vir}) \otimes_{U(\text{Der} \mathcal{O} \oplus \mathbb{C}C)} \mathbb{C}_c.
\]

Note that \( C \) acts on \( \text{Vir}_c \) by multiplication by \( c \). We say that \( \text{Vir}_c \) has central charge \( c \).

As before, the PBW theorem gives us an algebraic basis of \( \text{Vir}_c \). Let \( v_c \) be the image of \( 1 \otimes 1 \) in the tensor product definition of \( \text{Vir}_c \). Then the monomials

\[
L_{j_1} \cdots L_{j_m} v_c
\]
for \( j_1 \leq j_2 \leq \ldots \leq j_m < -1 \) form a basis for \( \text{Vir}_c \).

We now give \( \text{Vir}_c \) the structure of a vertex algebra, using the reconstruction theorem yet again:

- \( \deg v_c = 0, \deg L_n = -n \). This gradation corresponds to the eigenspace decomposition of \( \text{Vir}_c \) under the operator \( L_0 \). (This follows trivially from the formula for \([L_0, L_n]\).)
- \( T = L_{-1} \).
- \( Y(L_{-2}v_c, z) := T(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2} \).

Note that this time we are taking \( S \) to be a singleton; the single homogeneous vector here is \( L_{-2}v_c \). We verify the conditions of the reconstruction theorem. First, we need \( T(z) \) to be a field. This amounts to saying that for any \( w \in \text{Vir}_c \), \( L_n(w) = 0 \) for \( n \gg 0 \), which follows from the fact that \( L_n \) defines a degree \(-n\) operator on \( \text{Vir}_c \), and \( \text{Vir}_c \) has only non-negative degree components.

We verify the remaining conditions:

1. Note that \( |0\rangle \) is killed by \( L_n \) for all \( n \geq -1 \). So
   \[
   T(z)|0\rangle = \sum_{n < -1} L_n(|0\rangle)z^{-n-2} = \sum_{k \geq 0} L_{-2-k}v_c z^k = L_{-2}v_c + z(\ldots).
   \]

2. \( L_{-1}v_c = 0 \), by the definition of \( C_c \). We compute \([T, T(z)]\):
   \[
   [T, T(z)] = [L_{-1}, \sum_{n \in \mathbb{Z}} L_n z^{-n-2}] = \sum_{n \in \mathbb{Z}} [L_{-1}, L_n] z^{-n-2} = \sum_{n \in \mathbb{Z}} (-1 - n)L_{n-1}z^{-n-2} = \sum_{k \in \mathbb{Z}} (-k - 2)L_k z^{-k-3} = \partial_z T(z).
   \]

3. Yes, there is something to check here!! Namely, that \( T(z) \) must be local with respect to itself. It follows from the Lemma that \((z - w)^4[T(z), T(w)] = 0\).

4. This is the PBW basis we have already stated.

4. CONFORMAL VERTEX ALGEBRAS

(§2.5.7 of F&B)

Definition 7. A vertex algebra \( V \) is called conformal, of central charge \( c \in \mathbb{C} \), if we are given a nonzero vector \( \omega \in V_2 \) such that the Fourier coefficients \( L_n^V \) in

\[
Y(\omega, z) = \sum_{n \in \mathbb{Z}} L_n^V z^{-n-2}
\]
satisfy the defining relations of the Virasoro algebra with central charge \( c \), and in addition we have \( L_{-1}^V = T \), and \( L_0^V |_{V_n} = n \text{Id} \).

We refer to the vector \( \omega \) as a conformal vector.

In more modern terminology, we say that \( V \) is a Vertex Operator Algebra.

Note that in our standard notation for breaking up a field into its Fourier coefficients, \( \omega(n) = L_{n-1}^V \).

**Examples.**

- Obviously, the Virasoro algebra itself, with \( \omega = L_{-1} \).
- The Heisenberg vertex algebra \( \pi \) comes with a one-parameter family of conformal structures. For \( \lambda \in \mathbb{C} \), let
  \[
  \omega_{\lambda} = \frac{1}{2} b_{-1}^2 + \lambda b_{-2}.
  \]
  Then \( \omega_{\lambda} \) is a conformal vector with central charge \( c_{\lambda} = 1 - 12\lambda^2 \). There is a shortcut for verifying these relations, referenced later in the book.
- \( V_k(\mathfrak{g}) \) has a natural conformal vector, called the Sugawara vector, as long as \( k \neq -h^\vee \).

4.1. **The Sugawara Construction.**

(§2.5.10 F & B)

Let \( k \neq -h^\vee \) (where \( h^\vee \) is the dual Coxeter number of \( \mathfrak{g} \)).

Pick a basis \( J^1, \ldots, J^d \) for \( \mathfrak{g} \) and let \( J_1, \ldots, J_d \in \mathfrak{g} \) denote the dual basis with respect to the (non-degenerate) invariant form \((\cdot, \cdot)\) we constructed before.

Set
\[
S = \frac{1}{2} \sum_{a=1}^d J_{a-1} J_a^\vee v_k.
\]

This is independent of the choice of basis of \( \mathfrak{g} \).

Then
\[
\frac{S}{k + h^\vee}
\]
is a conformal vector in \( V_k(\mathfrak{g}) \), of central charge
\[
c(k) = \frac{k \cdot \text{dim} \mathfrak{g}}{k + h^\vee}.
\]

This is the Sugawara conformal vector.

5. **Vertex algebras associated to integral lattices**

5.1. **Modules over the Weyl algebra.**

(§4.2.1 of F&B)

Recall the Heisenberg algebra, \( \mathcal{H}' \), which has an algebraic basis given by \( b_n \), for \( n \in \mathbb{Z} \), and \( 1 \), with

- 1 central, and
- \( [b_n, b_m] = n \delta_{n,-m} 1 \).
We define the Weyl algebra \( \tilde{\mathcal{H}} \) as \( U(\mathcal{H}')/(1 - 1) \). (The book uses a completion of \( U(\mathcal{H}') \).)

That is, \( \tilde{\mathcal{H}} \) is the associative algebra with generators \( b_n \) for \( n \in \mathbb{Z} \), and relations \([b_n, b_m] = n\delta_{n,-m}\). This algebra is graded with \( \deg b_n = -n \).

Let \( \lambda \in \mathbb{C} \). We define the module \( \pi_\lambda \) to be generated by \( |\lambda\rangle \) such that

\[
b_n|\lambda\rangle = 0 \quad \forall n \geq 0
\]

and

\[
b_0|\lambda\rangle = \lambda|\lambda\rangle.
\]

Note that \( \lambda = 0 \) gives the Fock space representation we’ve seen before. Same as last week, we have that \( |\lambda\rangle \) freely generates \( \pi_\lambda \) as a module over the subalgebra \( \mathbb{C}[b_n]_{n<0} \subset \tilde{\mathcal{H}} \).

Under the identification \( \pi_\lambda = \mathbb{C}[b_n]_{n<0}|\lambda\rangle \), the action of \( b_n \) is given by:

- multiplication by \( b_n \) when \( n < 0 \),
- scalar multiplication by \( \lambda \), when \( n = 0 \), and
- \( n\delta/\partial b_{-n} \) when \( n > 0 \).

**Remark.** The presence of \( |\lambda\rangle \) serves to disambiguate when \( b_n \) is used to denote an element of \( \tilde{\mathcal{H}} \) and when it denotes an element of \( \pi_\lambda \), for \( n < 0 \).

We are interested in \( \mathbb{Z} \)-graded \( \tilde{\mathcal{H}} \)-modules with gradation bounded from below.

**Lemma 8.** Up to isomorphism, the irreducible \( \mathbb{Z} \)-graded \( \tilde{\mathcal{H}} \)-modules with gradation bounded from below are precisely \( \pi_\lambda \), for each \( \lambda \in \mathbb{C} \).

**Proof.** We show that \( \pi_\lambda \) is irreducible. Let \( x \in \pi_\lambda \). Then \( x = P|\lambda\rangle \) for some \( P \in \mathbb{C}[b_n]_{n<0} \). Selecting a monomial \( cb_{m_1}b_{m_2}\cdots b_{m_k} \) of maximal degree in \( P \), with \( m_1, \ldots, m_k > 0 \), we get that \( b_{m_1}b_{m_2}\cdots b_{m_k}x \) is a nonzero scalar multiple of \( |\lambda\rangle \). Then this vector generates \( \pi_\lambda \).

Conversely, let \( M \) be an irreducible graded \( \tilde{\mathcal{H}} \)-module. Let \( \tilde{M} \) be the lowest nontrivial homogeneous component of \( M \). Then \( b_0 \) stabilizes \( \tilde{M} \) and \( b_0\tilde{M} = 0 \) for \( n > 0 \). If \( \tilde{M}' \subset \tilde{M} \) is a proper subspace stable under \( b_0 \), then \( \tilde{\mathcal{H}}\tilde{M}' = \mathbb{C}[b_n]_{n<0}\tilde{M}' \) (because every element of \( \tilde{\mathcal{H}} \) can be written using monomials with the \( b_n \), \( n \leq 0 \) sorted before the others) is a proper submodule of \( M \). So \( \tilde{M} \) has no invariant subspaces under \( b_0 \). Since \( \mathbb{C} \) is algebraically closed, it follows that \( \dim \tilde{M} = 1 \), and \( b_0 \) acts on \( \tilde{M} \) by a scalar \( \lambda \). But then \( M \) is a quotient of \( \pi_\lambda \), therefore isomorphic to \( \pi_\lambda \). \( \square \)

5.2. Virasoro action and one-dimensional lattice vertex algebras.

(§4.2.4 of F&B)

The module \( \pi_\lambda \) carries an action of the Virasoro algebra, given by

\[
L_n = \frac{1}{2} \sum_{m \in \mathbb{Z}} :b_mb_{n-m}:
\]

for each \( n \in \mathbb{Z} \). This action has central charge 1.

Fix \( N \in 2\mathbb{Z}^>0 \) a positive even integer. Set

\[
V_{\sqrt{N}} := \bigoplus_{m \in \mathbb{Z}} \pi_{m\sqrt{N}}.
\]

**Proposition 9.** The module \( V_{\sqrt{N}} \) carries the structure of a vertex operator algebra such that \( \pi_0 = \pi \) is a vertex operator subalgebra of \( V_{\sqrt{N}} \).
Proof. (sketch)

We won’t work this out in detail, but a stronger version of the reconstruction theorem says it is enough to define the fields $Y(\langle \lambda \rangle, z)$, for each $\lambda \in \sqrt{N}\mathbb{Z}$. We will denote these fields by $V_{\lambda}(z) = Y(\langle \lambda \rangle, z)$.

Some work shows that defining these fields requires finding $c_{\lambda,\nu} \in \mathbb{C}$, for $\lambda, \nu \in \sqrt{N}\mathbb{Z}$, satisfying some equations.

Let $S_{\lambda} \in \text{End} V_{\sqrt{N}\mathbb{Z}}$ be a shift operator $\pi_{\mu} \rightarrow \pi_{\lambda+\mu}$, defined by

$$S_{\lambda}|\nu\rangle = c_{\lambda,\nu}|\nu+\lambda\rangle$$

and

$$[S_{\lambda}, b_n] = 0$$

for $n \neq 0$.

Then

$$V_{\lambda}(z) = S_{\lambda} z^{\lambda b_0} \exp \left( -\lambda \sum_{n<0} \frac{b_n}{n} z^{-n} \right) \exp \left( -\lambda \sum_{n>0} \frac{b_n}{n} z^{-n} \right).$$

Note that the exponential on the right is a locally finite sum, because $b_n = 0$ for $n \gg 0$ locally, so this product is well-defined.

Note also that the symbol $z^{\lambda b_0}$ requires that we utilize the diagonalization of $b_0$ on $V_{\sqrt{N}\mathbb{Z}}$ to make sense of it. In $\pi_{k\sqrt{N}}$, for $\lambda = m\sqrt{N}$, we have $z^{\lambda b_0} = z^{mkN}$.

Long story short, the equations on $c_{\lambda,\nu}$ boil down to:

$$c_{\lambda,0} = 1 \quad \forall \lambda \in \sqrt{N}\mathbb{Z}$$

(corresponding to the vacuum axiom) and

$$c_{\lambda,\nu} c_{\mu,\nu} = (-1)^{\lambda\mu} c_{\mu,\nu+\lambda} \quad \forall \lambda, \mu, \nu \in \sqrt{N}\mathbb{Z}$$

(corresponding to the locality axiom). Various solutions to these equations will give various different vertex operator algebra structures on $V_{\sqrt{N}\mathbb{Z}}$. □

6. Lattice vertex algebras

6.1. Heisenberg algebra associated to a lattice.

(§4.4.1 of F&B)

Let $L$ be a lattice of finite rank equipped with a symmetric bilinear form $(\cdot, \cdot) : L \times L \rightarrow \mathbb{Z}$ such that $(\lambda, \lambda) \in 2\mathbb{Z}_{>0}$ is a positive even integer for all $\lambda \in L - \{0\}$.

Set $\mathfrak{h} = L \otimes_{\mathbb{Z}} \mathbb{C}$, with a bilinear form induced from $L$. We can define a Heisenberg Lie algebra similarly to before, as a central extension of $\mathfrak{h}(\langle t \rangle)$, or of $\mathfrak{h}[[t^{\pm1}]]$ (which is what we will use).

We are interested in the corresponding Weyl algebra $\tilde{\mathcal{H}}_L$, obtained by identifying the center with the unit in the universal enveloping algebra; this algebra has generators $h_n$, for $h \in \mathfrak{h}$ and $n \in \mathbb{Z}$, together with the relations

$$[h_n, g_m] = n(h, g) \delta_{n,-m},$$

with the linear structure on the generators $\{h_n | h \in \mathfrak{h}\}$ for any fixed $n$ kept/induced from $\mathfrak{h}$. More straightforwardly, selecting a basis for $\mathfrak{h}$ gives generators and relations in the usual manner.

For each $\lambda \in \mathfrak{h}$, we define the Fock representation $\pi_\lambda$ of $\tilde{\mathcal{H}}_L$, generated by $|\lambda\rangle$ with relations

$$h_n |\lambda\rangle = 0$$

for $n \neq 0$. □
for \( n > 0 \) and
\[
h_0 |\lambda\rangle = (\lambda, h) |\lambda\rangle.
\]

The Fock representation \( \pi_0 \) carries a vertex algebra structure, similarly to the case we saw last week when \( \dim \mathfrak{h} = 1 \).


\((\S 4.4.2 \text{ of F&B})\)

Set
\[ V_L := \bigoplus_{\lambda \in L} \pi_{\lambda}. \]

Define a grading on \( V_L \) by \( \deg h_n = -n \), \( \deg |\lambda\rangle = (\lambda, \lambda)/2 \). This gradation takes only non-negative values.

There is a vertex operator algebra structure on \( V_L \) defined similarly to the case above when \( L = \sqrt{N} \mathbb{Z} \). There is a similarly formula for \( V_{\lambda}(z) \).

This requires finding \( c_{\lambda, \mu} \in \mathbb{C} \), for \( \lambda, \mu \in L \), satisfying the equations
\[
c_{\lambda, 0} = c_{0, \lambda} = 1, \quad (1)
\]
\[
c_{\lambda, \mu} = (-1)^{(\lambda, \mu)} c_{\mu, \lambda}, \quad (2)
\]
and
\[
c_{\mu, \nu} c_{\mu + \nu, \lambda} = c_{\mu, \nu + \lambda} c_{\nu, \lambda} \quad (3)
\]
for all \( \lambda, \mu, \nu \in L \).

6.3. The non-degenerate case.

\((\S 4.4.3 \text{ of F&B})\)

Now let’s assume \( c_{\lambda, \mu} \neq 0 \) for all \( \lambda, \mu \). Then equations \([1, 3]\) mean that \( \{c_{\lambda, \mu}\} : L \times L \to \mathbb{C}^\times \) is a two-cocycle of the additive group \( L \) with coefficients in \( \mathbb{C}^\times \).

Scaling the vectors \( |\lambda\rangle \) be constants \( a_\lambda \) transforms \( c_{\lambda, \mu} \) to \( a_\lambda a_\mu a_{\lambda + \mu}^{-1} c_{\lambda, \mu} \). This changes the two-cocycle by a coboundary. Thus finding solutions to equations \([1, 3]\) boils down to finding a cohomology class in \( H^2(L, \mathbb{C}^\times) \) satisfying the condition given by equation \([2]\). One can show that this cohomology class is unique for each isomorphism class of vertex operator algebra structure on \( V_L \).

The conformal vector of \( V_L \) is given by
\[
\omega = \frac{1}{2} \sum_{i=1}^{\dim \mathfrak{h}} (h_i)_- (h^i)_-, \quad \text{where} \quad \{h_i\} \text{ and } \{h^i\} \text{ are dual bases of } \mathfrak{h} \text{ with respect to } (\cdot, \cdot).
\]

Remark. If \( L \) is the root lattice of an ADE-type (also known as simply laced) simple Lie algebra, then \( V_L \) gives a construction of \( L(\Lambda_0) \), which we see next. This is the unique irreducible quotient of \( V_1(\mathfrak{g}) \), the vacuum representation of \( \hat{\mathfrak{g}} \) of level 1 (i.e., the case \( k = 1 \) above).
Let 
\[ g = \mathfrak{h} \oplus \left( \bigoplus_{\alpha \in \Delta} CE_{\alpha} \right) \]
be a simple finite-dimensional Lie algebra of type ADE. Let \((\cdot, \cdot)\) be the normalized invariant form on \(g\) that we have seen before. Let \(\widehat{g}\) be the corresponding affine Lie algebra, as a vector space \(\widehat{g} = g[t^{\pm 1}] \oplus \mathbb{C}K\). For \(u \in g\) and \(n \in \mathbb{Z}\), denote by \(u^{(n)} := t^n \otimes u\).

**Goal:** We will use vertex operator algebras to define the basic representation of \(\widehat{g}\), which is denoted by \(L(\Lambda_0)\). This representation is analogous to the defining representation of \(\mathfrak{sl}_n(\mathbb{C})\).

Let \(L\) denote the root lattice inside \(\mathfrak{h}\).

Consider the vector space \(V_L\). This vector space carries the action of \(\widehat{H}\), and comes equipped with the field from before, \(V_{\lambda}(z)\) for \(\lambda \in L\). We expand this field in terms of \(z\) in the familiar manner,

\[ V_{\lambda}(z) = \sum_{j \in \mathbb{Z}} V_{\lambda}^{(j)} z^{-j-1}. \]

**Theorem 10.** The map \(\sigma : \widehat{g} \rightarrow \text{End}(V)\) given by
\[
\begin{align*}
K & \mapsto 1, \\
\mathfrak{h} & \ni u^{(n)} \mapsto u_n, \text{ for } u \in \mathfrak{h}, n \in \mathbb{Z}, \\
CE_{\alpha}^{(n)} & \mapsto V_{\alpha}^{(n)}, \text{ for } \alpha \in \Delta, n \in \mathbb{Z},
\end{align*}
\]
defines the representation \(L(\Lambda_0)\).